INJECTIVE COVERS UNDER CHANGE OF RINGS

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Abstract. In [8], Würful gave a characterization of those rings $R$ which satisfy that for every ring extension $R \subset S$, $\text{Hom}_{R}(S, -)$ preserves injective envelopes. In this note, we consider an analogous problem concerning injective covers.

1. Introduction

Let $R$ be a ring with identity 1 and let every module be unitary. We will use the terminology of Enochs [2].

An injective cover of an $R$-module $M$ is a linear map $\phi : E \to M$ with an injective $R$-module $E$ such that

(1) for any injective $R$-module $E'$ and any linear map $\phi' : E' \to M$, the diagram

\[
\begin{array}{ccc}
E' & \longrightarrow & M \\
\downarrow \phi' & & \downarrow \\
E & \xrightarrow{\phi} & M
\end{array}
\]

can be completed to a commutative diagram.

(2) the diagram

\[
\begin{array}{ccc}
E & \longrightarrow & M \\
\downarrow \phi & & \\
E & \xrightarrow{\phi} & M
\end{array}
\]

can only be completed by automorphism of $E$.

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Hence if an injective cover exists, it is unique up to isomorphism. If \( \phi : E \rightarrow M \) satisfies (1), and perhaps not (2), it is called an injective precover. We will sometimes simply say \( E \) is an injective cover (or precover).

The existence of an injective cover is not guaranteed for all cases but every \( R \)-module has an injective cover if and only if the ring \( R \) is Noetherian (see [2, Theorem 2.1]). However, examples of injective covers are hard to come by. The first nontrivial example was constructed by Cheatham, Enochs, and Jenda [1] when \( R = \kappa[x_1, x_2, \cdots, x_n] \), \( n \geq 2 \), where \( \kappa \) is a field. In this case, let \( \mathcal{P} = (x_1, x_2, \cdots, x_n) \), \( R/\mathcal{P} = \kappa \) (with \( x_i \kappa = 0 \) for \( i = 1, 2, \cdots, n \)) and let \( E(\kappa) \) denote the injective envelope of \( \kappa \). Then the natural map \( E(\kappa) \rightarrow E(\kappa)/\kappa \) is an injective cover. This used Northcott’s description [5] of \( E(\kappa) \) as the inverse polynomial ring \( \kappa[x_1^{-1}, x_2^{-1}, \cdots, x_n^{-1}] \). Another example is when \( R \) is an \( n \)-dimensional regular local ring with residue field \( \kappa \). If \( n \geq 2 \), then again the natural map \( E(\kappa) \rightarrow E(\kappa)/\kappa \) is an injective cover (see [3, Corollary 4.2]).

**Lemma 1.1.** (Wakamatsu, [7]; [9, Lemma 2.1.1]) Let \( \phi : E \rightarrow M \) be an injective cover of an \( R \)-module \( M \). Then \( \ker \phi \) has the property that \( \text{Ext}_R^1(\bar{E}, \ker \phi) = 0 \) for any injective \( R \)-module \( \bar{E} \).

**Definition 1.2.** A special injective precover is defined to be a precover \( \phi : E \rightarrow M \) such that \( \ker \phi \) has the property that \( \text{Ext}_R^1(\bar{E}, \ker \phi) = 0 \) for any injective \( R \)-module \( \bar{E} \).

**Proposition 1.3.** (Kim, Park, Song [4, Proposition 1.3]) If an \( R \)-module \( M \) has an injective cover and \( \phi : E \rightarrow M \) is an injective precover of \( M \), then the followings are equivalent;

(a) \( \phi \) is an injective cover of \( M \)
(b) There is no nonzero direct summand of \( E \) contained in \( \ker \phi \)
(c) Any linear map \( f : E \rightarrow E \) with \( \phi \circ f = \phi \) is a surjection.

**2. Ring extensions and injective precovers**

In [8], Würful gave a characterization of those rings \( R \) such that for every ring extension \( R \subset S \), \( \text{Hom}_R(S, -) \) converts injective envelopes of \( R \)-modules into injective envelopes of \( S \)-modules. In this section, we will consider an analogous problem concerning injective covers.
Lemma 2.1. Let \( f : R \to S \) be a ring homomorphism and \( S_R \) flat. If \( sE \) is injective, then \( R_E \) is also injective.

Proof. Let \( g : I \to_R E \) be an \( R \)-linear map for an ideal \( I \) of \( R \). Define \( \alpha : S \otimes_R I \to E \) by \( \alpha(s \otimes x) = sg(x) \). Then \( \alpha \) is \( S \)-linear. Also \( 0 \to S \otimes_R I \to S \otimes_R R \) is exact since \( S_R \) is flat. So the diagram

\[
\begin{array}{c}
S \otimes_R I \to S \otimes_R R \\
\downarrow \alpha \\
E
\end{array}
\]

can be completed to a commutative diagram, where \( \iota : I \to R \) is the inclusion map. But the composition map \( I \to S \otimes_R I \to E \) defined by \( \beta(a) = 1 \otimes a \) is equal to the original \( g \). So \( R \to S \otimes_R R \to E \) gives an \( R \)-linear extension.

Remark 2.2. Let \( f : R \to S \) be a ring homomorphism and let \( E \) be an injective \( R \)-module. Then for any \( S \)-module \( M \), \( \text{Ext}^1_S(M, \text{Hom}_R(S, E)) \cong \text{Ext}^1_R(S \otimes M, E) = 0 \), and thus \( \text{Hom}_R(S, E) \) is an injective \( S \)-module.

Theorem 2.3. Let \( f : R \to S \) be a ring homomorphism, \( S_R \) flat and \( \phi : E \to M \) be an injective precover of an \( R \)-module \( M \). Then \( \text{Hom}_R(S, E) \to \text{Hom}_R(S, M) \) is a special injective precover.

Proof. To show that \( \text{Hom}_R(S, E) \to \text{Hom}_R(S, M) \) is an injective precover, it suffices to show that

\[ \text{Hom}_S(\bar{E}, \text{Hom}_R(S, E)) \to \text{Hom}_S(\bar{E}, \text{Hom}_R(S, M)) \to 0 \]

is exact for any injective \( S \)-module \( \bar{E} \), or equivalently to show that \( \text{Hom}_R(S \otimes S \bar{E}, E) \to \text{Hom}_R(S \otimes S \bar{E}, M) \to 0 \) is exact. Since \( \phi : E \to M \) is an injective precover of \( M \) and \( S \otimes_S \bar{E} \cong \bar{E} \) is \( R \)-injective by Lemma 2.1, therefore \( \text{Hom}_R(S \otimes_S \bar{E}, E) \to \text{Hom}_R(S \otimes_S \bar{E}, M) \to 0 \) is exact.

Next we need to show that \( \text{Hom}_R(S, \text{Ker}\phi) \) has the property that for any injective \( S \)-module \( E' \), \( \text{Ext}_S^1(E', \text{Hom}_R(S, \text{Ker}\phi)) = 0 \).
Since $\text{Ker}(\text{Hom}_R(S, E) \to \text{Hom}_R(S, M)) \cong \text{Hom}_R(S, \text{Ker}\phi)$,
\[
\text{Ext}_S^1(E', \text{Hom}_R(S, \text{Ker}\phi)) \cong \text{Ext}_R^1(S \otimes_S E', \text{Ker}\phi)
\cong \text{Ext}_R^1(E', \text{Ker}\phi) = 0.
\]

**Corollary 2.4.** With the above situations, the followings are equivalent;

(1) $\psi : \text{Hom}_R(S, E) \to \text{Hom}_R(S, M)$ is an injective cover

(2) (a) $\phi : E \to M$ is an injective precover

(b) $\text{Hom}_R(S, \text{Ker}\phi)$ has no nonzero injective submodules in $\text{Hom}_R(S, E)$

(3) $\psi$ is an injective precover and $\text{Hom}_R(S, \text{Ker}\phi)$ has no nonzero injective submodules in $\text{Hom}_R(S, E)$.

**Example 2.5.** Let $S = R[x]$. Given an injective cover $\phi : E \to M$, $\text{Hom}_R(R[x], E) \to \text{Hom}_R(R[x], M)$ is a special injective precover since $R[x]$ is a flat $R$-module. Note that $\text{Hom}_R(R[x], E) \cong E[[x^{-1}]]$ and $\text{Hom}_R(R[x], M) \cong M[[x^{-1}]]$. Since $\phi : E \to M$ is an injective cover, $K = \text{Ker}\phi$ has no nonzero injective submodules. So $E[[x^{-1}]] \to M[[x^{-1}]]$ is an injective cover if $K[[x^{-1}]]$ has no nonzero injective submodule as $R[x]$-module. But any injective $R[x]$-module is injective as an $R$-module. So $E[[x^{-1}]] \to M[[x^{-1}]]$ is an injective cover if $K[[x^{-1}]] \cong K \times K \times K \times \cdots$ has no nonzero injective submodule as an $R$-module.

**Proposition 2.6.** Let $R$ be a semi-local ring and $\phi : E \to M$ an injective cover. Then

(1) $\text{Ext}_R^n(\bar{E}, \text{Ker}\phi) = 0$ for all $n > 1$ and injective $\bar{E}$.

(2) $\text{Ext}_R^n(\bar{E}, E) \cong \text{Ext}_R^n(\bar{E}, M)$ for all injective $\bar{E}$ and $n \geq 1$.

**Proof.** (1) For any injective $R$-module $\bar{E}$, let $0 \to K \to F \to \bar{E} \to 0$ be an exact sequence with $F$ free. Since $\text{Ext}_R^n(F, \text{Ker}\phi) = 0$ for all $n \geq 1$, $\text{Ext}_R^n(K, \text{Ker}\phi) \cong \text{Ext}_R^{n+1}(\bar{E}, \text{Ker}\phi)$ for all $n \geq 1$. And since $K$ is injective, $\text{Ext}_R^n(K, \text{Ker}\phi) = 0$. So $\text{Ext}_R^n(\bar{E}, \text{Ker}\phi) = 0$. Proceeding in this manner, $\text{Ext}_R^n(\bar{E}, \text{Ker}\phi) = 0$ for all $n \geq 1$.

(2) It follows from $\text{Hom}_R(\bar{E}, E) \to \text{Hom}_R(\bar{E}, M) \to 0$ is exact for all injective $\bar{E}$ and $\text{Ext}_R^n(\bar{E}, \text{Ker}\phi) = 0$ for all $n \geq 1$. \qed
Remark 2.7. Suppose that for any module $M$ over a ring $R$, $\text{Ext}^1_R(E, M) = 0$ for all injective $R$-module $E$ implies that $\text{Ext}^i_R(E, M) = 0$ for all injective $R$-module $E$ and $i \geq 1$. If $\text{inj.dim}_R M = n < \infty$, then $n = 0$, i.e. $M$ is injective. For if $0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots \rightarrow E^n \rightarrow 0$ is an injective resolution of $M$ with $n \geq 1$, then $\text{Ext}^n_R(E^n, M) = 0$. This means

\[
\begin{array}{ccc}
E^n & \downarrow \text{id} \\
E^{n-1} & \rightarrow & E^n \\
\end{array}
\]

can be completed to a commutative diagram. But then $E^{n-1} \cong E \oplus E^n$ for some injective $E$ and we have an injective resolution $0 \rightarrow M \rightarrow E^0 \rightarrow \cdots \rightarrow E^{n-2} \rightarrow E \rightarrow 0$ of $M$ of length $n - 1$. If $n - 1 \geq 1$, then we can repeat the procedure.

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