ON THE WEAK INVARIANCE PRINCIPLE FOR RANGES OF RECURRENT RANDOM WALKS WITH INFINITE VARIANCE

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1. Introduction

Let \( \{ X_k : k = 1, 2, \ldots \} \) be a sequence of independent, identically distributed integer-valued random variables with common distribution function \( F \). Throughout this paper we assume that

(A1) \( F \) belongs to the domain of attraction of a strictly \( \alpha \)-stable distribution with \( 1 < \alpha \leq 2 \),

(A2) \( E X_1 = 0 \),

(A3) \( E \exp(iuX_1) = 1 \) if and only if \( u \) is a multiple of \( 2\pi \).

We note that \( \{ S_n \} \) is an aperiodic recurrent random walk, where \( S_0 = 0 \) and \( S_n = \sum_{k=1}^n X_k \). Let \( \varphi(u) = E \exp(iuX_1) \). Then it is well-known that

\[
|\varphi(u)| = \exp \{-|u|^{\alpha}l(1/|u|)\} \quad \text{for} \quad |u| \leq \pi,
\]

where \( l(x) \) is a slowly varying function at infinity. Furthermore if we choose \( a_n \) so that

\[
\frac{l(a_n)}{a_n^{\alpha}} = \frac{1}{n}
\]

for each \( n \), then \( Y^{(n)}(t) = S_{\lfloor nt \rfloor}/a_n \) converges weakly to a strictly \( \alpha \)-stable process \( Y(t) \), where \( \lfloor x \rfloor \) denotes the greatest integer not exceeding \( x \) (e.g. see page 345 of [1]).

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The range \( R_n \) of random walk \( \{ S_k \} \) and the range \( \Lambda(t) \) of stable process \( Y \) are defined as follows;

\[
R_n = \text{the cardinality of } \{S_0, S_1, \cdots, S_n\}
\]

and

\[
\Lambda(t) = m\{Y(s) : 0 \leq s \leq t\},
\]

where “\( m \)” denotes the Lebesgue measure on \( \mathbb{R}^1 \). Set \( \Lambda^{(n)}(t) = R_{[nt]}/a_n \).

The aim of the present work is to prove weak convergence of \( \Lambda^{(n)}(t) \) to \( \Lambda \). In fact, we obtain the existence of \( \tilde{\Lambda}^{(n)} \) and \( \tilde{\Lambda} \), versions of \( \Lambda^{(n)} \) and \( \Lambda \), respectively, such that \( \tilde{\Lambda}^{(n)}(t) \) converges to \( \tilde{\Lambda}(t) \) uniformly on \([0, T]\) in \( L^2 \)-sense for any \( T > 0 \).

Le Gall and Rosen [6] obtained various limit theorems for the range of \( d \)-dimensional random walk in the domain of attraction of a stable distribution of index \( \alpha \). Their results depend on the value of the ratio \( \alpha/d \). That is, for the case \( \alpha/d \leq 1 \), strong law of large numbers and central limit theorems hold and for the case \( \alpha > d = 1 \) which we are concerned with in this work, \( R_n/a_n \) converges in distribution to \( \Lambda(1) \). In this work, we extend their result and prove the weak convergence of \( \Lambda^{(n)} \) to \( \Lambda \).

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Borodin [3] obtained a weaker result for the similar question for recurrent random walks with finite variance.

Now we state our main result, whose proof is given in Section 2.

**Theorem.** Under the assumptions (A1), (A2) and (A3), there exist processes \( \tilde{Y}^{(n)} \) and \( \tilde{Y} \) in \( D[0, \infty) \) equipped with Skorokhod metric satisfying the following conditions:

(i) \( \tilde{Y}^{(n)} \equiv D Y^{(n)} \), \( \tilde{Y} \equiv D Y \),

(ii) \( \tilde{Y}^{(n)} \) converges to \( \tilde{Y} \) a.s. in \( D[0, \infty) \), and

(iii) for each \( T > 0 \) and positive integer \( m \),

\[
E \left[ \sup_{0 \leq t \leq T} \left| \tilde{\Lambda}^{(n)}(t) - \tilde{\Lambda}(t) \right|^{2m} \right] \longrightarrow 0 \quad \text{as } n \rightarrow \infty,
\]

where \( \tilde{\Lambda}^{(n)} \) and \( \tilde{\Lambda} \) are defined with respect to \( \tilde{Y}^{(n)} \) and \( \tilde{Y} \), respectively, and “\( \equiv D \)” means that two processes have the same finite dimensional distributions.
2. Proof of Main Result

Recall that we assume (A1), (A2) and (A3). Throughout this work, we denote $P_0$ and $E_0$ by $P$ and $E$, respectively.

We present the proof of the Theorem in this section. Since the construction of $\tilde{Y}^{(n)}$ and $\tilde{Y}$ satisfying parts (i) and (ii) of the Theorem are well-known (e.g. see chapter 1 of [7]), it suffices to establish part (iii) of the Theorem. Therefore we may abuse our notation and use $Y^{(n)}, Y, \Lambda^{(n)}$ and $\Lambda$ for $\tilde{Y}^{(n)}, \tilde{Y}, \tilde{\Lambda}^{(n)}$ and $\tilde{\Lambda}$, respectively throughout the remainder of the work. Essentially, the proof of our assertion amounts to estimating

\begin{equation}
E \left[ (\Lambda^{(n)}(t) - \Lambda(t))^{2m} \right],
\end{equation}

since a simple monotonicity argument using also continuity of $\Lambda(t)$ implies our assertion if (2.1) converges to zero. Le Gall and Rosen [6] showed that $\tilde{\Lambda}^{(n)}(t)$ converges to $\tilde{\Lambda}(t)$ in $L^1$-sense, but their technique doesn’t work in general. To deal with (2.1), we express the ranges of random walks and stable processes using their local times, respectively. The local time $N(n, x)$ of random walk $\{S_k\}$ is defined by

\[ N(n, x) = \text{the number of } \{ 0 \leq k \leq n : S_k = x \}. \]

Let

\[ L^{(n)}(t, x) = \frac{a_n}{n} N([nt], [xa_n]) \]

and

\[ W_n(t) = \{ x \in \mathbb{R} : L^{(n)}(t, x) > 0 \}. \]

Then we note that

\begin{align*}
\Lambda^{(n)}(t) &= \frac{1}{a_n} \sum_{k \in \mathbb{Z}} \chi_{[N([nt], k) > 0]} \\
&= \int_{\mathbb{R}} \chi_{W_n(t)}(x) \, dx.
\end{align*}

For a stable process $Y(t)$ of index $1 < \alpha < 2$, it is well-known that there exists a version of local time $\{L(t, x)\}$ which is jointly continuous in $(t, x)$
and satisfies the so-called occupation time density formula, that is, for any Borel set $B$,
$$
\int_B L(t, x) \, dx = \int_0^t \chi_B(Y(s)) \, ds \quad \text{a.s.}
$$
The existence and joint continuity of $L(t, x)$ were proved by Trotter [8] for Brownian motion and by Boylan [4] for stable processes of index $\alpha > 1$. Moreover, Kang and Wee [5] proved that as $n \to \infty$,
\begin{equation}
(2.2) \quad \sup_{(t, x) \in [0, T] \times \mathbb{R}^1} \left| L^{(n)}(t, x) - L(t, x) \right| \to 0 \quad \text{in } L^2.
\end{equation}
We provide an useful expression of $\Lambda(t)$ in terms of local time $L(t, x)$ in Lemma 2.1, and then apply the result of [5] to estimate (2.1).

**Lemma 2.1.** For each $t \geq 0$,
$$
\Lambda(t) = \int_{\mathbb{R}^1} \chi_{W(t)}(x) \, dx \quad \text{a.s.}
$$
where $W(t) = \{ x \in \mathbb{R}^1 : L(t, x) > 0 \}$.

**Proof.** We write
$$
\Lambda(t) = m(G(t)) + m(W(t)),
$$
where
$$
G(t) = \{ x \in \mathbb{R}^1 : Y(s) = x \text{ for some } s \in [0, t], \ L(t, x) = 0 \}.
$$
Let
$$
\tau_x = \inf\{ s \geq 0 : Y(s) = x \},
$$
\$$
\hat{Y}(s) = Y(s + \tau_x) - x,
$$
and $\hat{L}(s, y)$ be the local time of $\hat{Y}$. The strong Markov property implies that
\begin{align}
E[m(G(t))] &= \int_{\mathbb{R}^1} P\left( \tau_x \leq t, \ \hat{L}(t - \tau_x, 0) = 0 \right) \, dx \\
&= \int_{\mathbb{R}^1} \int_0^t P\left( \hat{L}(t - s, 0) = 0 \right) P(\tau_x \in ds) \, dx \\
&= 0,
\end{align}
where the last equality follows from the definition of $\hat{L}(t, 0)$ as a continuous additive functional with support $\{0\}$ (see page 216 of [2]).
LEMMA 2.2. For each $t \geq 0$ and positive integer $m$,

$$E \left[ \left( \Lambda^{(n)}(t) - \Lambda(t) \right)^{2m} \right] \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$ 

Proof. Recall that

$$W_n(t) = \{ x \in \mathbb{R}^1 : L^{(n)}(t,x) > 0 \}$$

and

$$W(t) = \{ x \in \mathbb{R}^1 : L(t,x) > 0 \}.$$ 

For each $K > 0$, let

$$\Lambda^{(n)}_K(t) = \int_{-K}^{K} \chi_{W_n(t)}(x) \, dx$$

and

$$\Lambda_K(t) = \int_{-K}^{K} \chi_{W(t)}(x) \, dx.$$ 

Then

$$E \left[ \left( \Lambda^{(n)}(t) - \Lambda^{(n)}_K(t) \right)^{2m} \right] \leq E \left[ \Lambda^{(n)}(t)^{2m} \cdot \chi_{\{ \sup_{0 \leq l \leq [nt]} |S_l| > Ka_n \}} \right]$$

$$\leq \left\{ E \left[ (\Lambda^{(n)}(t))^{4m} \right] \right\}^{1/2} \cdot \left\{ P \left( \sup_{0 \leq l \leq [nt]} |S_l| > Ka_n \right) \right\}^{1/2}.$$ 

Weak convergence of $Y^{(n)}$ to $Y$ implies that for any $\varepsilon > 0$, there exists $K = K(\varepsilon)$ such that

$$\sup_n P \left( \sup_{0 \leq l \leq [nt]} |S_l| > Ka_n \right) < \varepsilon.$$ 

It follows from [6] that any finite moment of $\Lambda^{(n)}(t)$ is bounded uniformly in $n$. Thus by (2.5), we may choose $K$ large so that (2.4) is sufficiently small for all $n$ large. For $1 < \alpha < 2$, it is easy to see that for any $u > 0$,

$$E \left[ \exp(u \Lambda(t)) \right] < \infty$$ 

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without assuming the symmetry of \(Y(t)\), by modifying the argument in Lemma 4.1 of [9]. Thus
\[
E \left[ \Lambda(t)^{2m} \right] < \infty,
\]
which is obvious for \(\alpha = 2\). This enables us to have that for any \(\varepsilon > 0\), there exists \(K = K(\varepsilon)\) such that
\[
E \left[ (\Lambda(t) - \Lambda_K(t))^{2m} \right] < \varepsilon.
\]

Now we fix \(K\) large enough, and observe that

\[
E \left[ \left( \int_{-K}^{K} \chi_{W_{\alpha}(t) \cap W(t)}(x) \, dx - \int_{-K}^{K} \chi_{W_{\alpha}(t) \cap W(t)}(x) \, dx \right)^{2m} \right]
\]

(2.6)

\[
\leq 2^{4m-2} K^{2m-1} E \left[ \int_{-K}^{K} \chi_{W_{\alpha}(t) \cap W(t)'}(x) \, dx \right]
+ 2^{4m-2} K^{2m-1} E \left[ \int_{-K}^{K} \chi_{W_{\alpha}(t) \cap W(t)}(x) \, dx \right].
\]

Let \(Y[0; t] = \{Y(s) : 0 \leq s \leq t\}\), \(cl(Y[0; t])\) be its closure, and \(U_{\delta}(t)\) be the \(\delta\)-neighborhood of \(cl(Y[0; t])\). Then as \(\delta \to 0\),
\[
m(U_{\delta}(t)) \to m(cl(Y[0; t])) = m(Y[0; t]) \quad \text{a.s.}
\]

and part (ii) of the Theorem implies that for fixed \(\delta > 0\),

\[
W_n(t) \subset U_{\delta}(t) \quad \text{a.s.}
\]

(2.7)

for all sufficiently large \(n\). Now fix \(\delta > 0\) so that \(E \left[ m \left( U_{\delta}(t) \cap Y[0; t]^c \right) \right]\) is sufficiently small. Then by (2.7) and (2.3), for \(n\) large,
\[
E \left[ \int_{-K}^{K} \chi_{W_{\alpha}(t) \cap W(t)'}(x) \, dx \right]
\]

\[
\leq E \left[ \int_{-K}^{K} \chi_{U_{\delta}(t) \cap W(t)'}(x) \, dx \right]
\]

\[
\leq E \left[ \int_{-K}^{K} \chi_{Y[0; t] \cap W(t)'}(x) \, dx \right] + E \left[ m \left( U_{\delta}(t) \cap Y[0; t]^c \right) \right]
\]

\[
\leq E \left[ m \left( G(t) \right) \right] + E \left[ m \left( U_{\delta}(t) \cap Y[0; t]^c \right) \right]
\]

\[= E \left[ m \left( U_{\delta}(t) \cap Y[0; t]^c \right) \right].
\]
Hence the first summand of (2.6) can be made arbitrarily small for \( n \) sufficiently large. To estimate the second summand of (2.6), observe that for any \( \eta > 0 \),

\[
P \left( L(t, x) > 0, \ L^{(n)}(t, x) = 0 \right) \leq P \left( 0 < L(t, x) < \eta \right) + P \left( \left| L^{(n)}(t, x) - L(t, x) \right| \geq \eta \right).
\]

Use (2.2) and bounded convergence theorem to show that

\[
E \left[ \int_{-K}^{K} \chi_{W(t) \cap W_{n}(x)}(x) \, dx \right] = \int_{-K}^{K} P \left( L(t, x) > 0, \ L^{(n)}(t, x) = 0 \right) \, dx \to 0
\]
as \( n \) goes to infinity.

**Proof of the Theorem.** Fix \( h > 0 \), which will be chosen later. Let \( 0 = t_0 < t_1 < \cdots < t_k \leq t_{k+1} = T \) be a partition of \([0, T]\) such that \( t_j - t_{j-1} = h \) for all \( 1 \leq j \leq k \) and \( k = \lfloor T/h \rfloor \). Observe that by simple monotonicity of \( \Lambda^{(n)} \) and \( \Lambda \),

\[
E \left[ \sup_{0 \leq t \leq T} \left| \Lambda^{(n)}(t) - \Lambda(t) \right|^{2m} \right]
\]

\[
\leq 3^{2^{m-1}} 2^{2m} E \left[ \max_{0 \leq j \leq k} \left| \Lambda(t_{j+1}) - \Lambda(t_j) \right|^{2} \right]
\]

\[
+ 3^{2^{m-1}} (2^{2m} + 1) \sum_{j=0}^{k+1} E \left[ \left| \Lambda^{(n)}(t_j) - \Lambda(t_j) \right|^{2} \right].
\]

By the almost sure continuity of the mapping \( t \mapsto \Lambda(t) \) and dominate convergence theorem, for given \( \varepsilon > 0 \), we can choose \( h \) so that the first term of (2.8) is less than \( \varepsilon/2 \). Then Lemma 2.2 implies that the second term of (2.8) can be made arbitrarily small if \( n \) is large enough, which completes the proof.
References


On the weak invariance principle

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