HOMOLOGY OF THE TRIPLE LOOP SPACE
OF THE EXCEPTIONAL LIE GROUP $F_4$

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Abstract. We study the homology of the triple loop space of the exceptional Lie group $F_4$ by exploiting the spectral sequences and the homology operations.

Introduction

The homology of the iterated loop space is one of the main topics in topology. Besides its own interest it has many applications to other branches. Especially the homology of the triple loop space is very interesting from the gauge theoretic viewpoint.

Let $G$ be a compact connected simple Lie group. Since $\pi_3(G) = \pi_4(BG) = \mathbb{Z}$, the principal $G$ bundles $P_k$ over $S^4$ are classified by the integer $k$ in $\mathbb{Z}$. For a given $P_k$, the orbit space of connections up to the based gauge equivalence is homotopy equivalent to the triple loop space of $G$ [1]. Then there is a natural inclusion map $i: M_k \to C_k \simeq \Omega^3 G$ where $M_k$ is the moduli space of $G$ instantons. Moreover the inclusion map $i: M_\infty \to C_\infty$ induces a homotopy equivalence [11] where $M_\infty$ and $C_\infty$ are the direct limits under the inclusions. So the homology of the triple loop space is a cornerstone for getting information about the homology of the instanton space [2], [3], [4].

In this paper we study the homology of the triple loop space of the exceptional Lie group $F_4$ by computing the Eilenberg–Moore spectral sequence and the Serre spectral sequence with the aid of the Dyer–Lashof operations.

$G$ is called $p$-regular if $G$ is $p$-equivalent to a product of spheres, i.e.,

$$G \simeq_p \prod S^{2m_i+1}$$
and $G$ is called quasi $p$-regular if $G$ is $p$-equivalent to a product of spheres and sphere bundles over spheres of type $B(2n+1, 2n+2p-1)$, i.e.,

$$G \simeq_p \prod S^{2m_i+1} \times \prod B(2n_j + 1, 2n_j + 2p - 1).$$

Here $B(2n+1, 2n+2p-1)$ is a mod $p$ H-space defined as an $S^{2n+1}$-bundle over $S^{2n+1+2(p-1)}$ with characteristic element $\alpha_p$, where $\alpha_p$ is the generator of the $p$-primary component of $\pi_{2n+2(p-1)}(S^{2n})$.

It is well-known that $F_4$ is $p$-regular if and only if $p \geq 13$ and $F_4$ is quasi $p$-regular if and only if $p \geq 5$ [10]. Hence for $p \geq 5$, the homology of the triple loop space of $F_4$ is simply determined by the homologies of the triple loop spaces of spheres and the homologies of the triple loop spaces of $B(2n_j + 1, 2n_j + 2p - 1)$'s. So we first determine the case $p \geq 5$ and then we concentrate on the cases $p = 2$ and $p = 3$.

1. Preliminaries

Let $E(x)$ be the exterior algebra on $x$ and $\Gamma(x)$ be the divided power algebra on $x$ which is free over $\gamma_i(x)$ as a $\mathbb{F}_p$ module with the product $\gamma_i(x)\gamma_j(x) = \binom{i+j}{i}\gamma_{i+j}(x)$. In this paper the subscript of the element always denotes the degree of that element.

We have homology operations $Q_i$ on the $(n+1)$-loop space $\Omega^{n+1}X$

$$Q_{i(p-1)} : H_q(\Omega^{n+1}X; \mathbb{F}_p) \to H_{pq+i(p-1)}(\Omega^{n+1}X; \mathbb{F}_p)$$

for $0 \leq i \leq n$ which are natural with respect to $(n+1)$-loop maps. In particular, we have $Q_0x = x^p$. The iterated power $Q^a_i$ denotes the composition of $Q_i$’s $a$ times, i.e., $Q^a_i = \underbrace{Q_i \circ \cdots \circ Q_i}_{a \text{ times}}$.

These operations satisfy the following properties.

**Proposition 1.1.** [6] In the path-loop fibration

$$\Omega^{n+2}X \to P\Omega^{n+1}X \to \Omega^{n+1}X,$$

we have the following:

1. if $x \in H_q(\Omega^{n+1}X; \mathbb{F}_p)$ is transgressive in the Serre spectral sequence, then so is $Q_i x$ and $\tau \circ Q_{i(p-1)}x = Q_{(i+1)(p-1)} \circ \tau x$ for each $i$, $0 \leq i \leq n$ where $\tau$ is the transgression,
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(2) for $p > 2$ and $n \geq 1$, $d^{i |(p-1)}(x^{p-1} \otimes \tau(x)) = -\beta Q_{(p-1)} \tau(x)$,

(3) for $p = 2$, $Sq^1 Q;x = Q_{i-1}x$ if $x \in H_q(\Omega^{n+1} X; \mathbb{F}_2)$ and $q + i$ is even.

Let $P^*(\Omega X; \mathbb{F}_p)$ be the primitives of $H^*(\Omega X; \mathbb{F}_p)$ and $Q^*(\Omega X; \mathbb{F}_p)$ be the indecomposables of $H^*(\Omega X; \mathbb{F}_p)$.

**PROPOSITION 1.2.** [7] Let $X$ be a simply connected $H$-space. Then the following is true:

1. The suspension $\sigma : Q_{odd}^*(X; \mathbb{F}_p) \rightarrow P_{even}^*(\Omega X; \mathbb{F}_p)$ is injective,
2. The suspension $\sigma : Q_{even}^*(X; \mathbb{F}_p) \rightarrow P_{odd}^*(\Omega X; \mathbb{F}_p)$ is onto,
3. The quotient $P_{even}^*(\Omega X; \mathbb{F}_p)/\sigma(Q_{odd}^*(X; \mathbb{F}_p))$ is obtained by transpotence, and
4. The elements in $\ker \sigma$ are dual to elements in the image of the homology transpotence.

For $n > 1$, as an algebra the homology of the triple loop space of the odd sphere is determined by [6]

$$
H_*^{\mathbb{F}_p}(\Omega^3 S^{2n+1}; \mathbb{F}_2) = \mathbb{F}_2[Q_1^a Q_2^b u_{2n-2} : a \geq 0, b \geq 0],
$$

$$
H_*^{\mathbb{F}_p}(\Omega^3 S^{2n+1}; \mathbb{F}_p) = \mathbb{F}_p[Q_{2(p-1)}^a u_{2n-2} : a \geq 0]
$$

$$
\otimes E(Q_{(p-1)}^a \beta Q_{2(p-1)}^b u_{2n-2} : a \geq 0, b > 0)
$$

$$
\otimes \mathbb{F}_p[\beta Q_{(p-1)}^a \beta Q_{2(p-1)}^b u_{2n-2} : a > 0, b > 0], p: \text{ odd.}
$$

The exceptional Lie group $F_4$ when localized at $p$ splits as follows:

$$F_4 \simeq (p) B(3, 11) \times B(15, 23), \quad p = 5,$$

$$F_4 \simeq (p) B(3, 15) \times B(11, 23), \quad p = 7,$$

$$F_4 \simeq (p) B(3, 23) \times S^{11} \times S^{15}, \quad p = 11,$$

$$F_4 \simeq (p) \Pi_{k=0}^1 S^{8k+3} \times \Pi_{k=0}^1 S^{8k+15}, \quad p \geq 13.$$

Hence $H_*^{\mathbb{F}_p}(\Omega^3 F_4; \mathbb{F}_p)$ is a tensor product of $H_*^{\mathbb{F}_p}(\Omega^3 S^{2n+1}; \mathbb{F}_p)$’s for $p \geq 13$, and $H_*^{\mathbb{F}_p}(\Omega^3 F_4; \mathbb{F}_p)$ a tensor product of $H_*^{\mathbb{F}_p}(\Omega^3 S^{2n+1}; \mathbb{F}_p)$’s and $H_*^{\mathbb{F}_p}(\Omega^3 B(2n+1, 2n+2p-1); \mathbb{F}_p)$’s for $p \geq 5$.

The space $B(2n+1, 2n+2p-1)$ is equivalent to a direct factor of the $p$–localization of $SU(n+p)/SU(n)$ [10] and its cohomology ring is

$$H^*(B(2n+1, 2n+2p-1); \mathbb{F}_p) = E(x_{2n+1}, x_{2n+2p-1})$$

with $P^1 x_{2n+1} = x_{2n+2p-1}$.

So from the computation of the homology of the triple loop space of the special unitary group $SU(m)$[4, 12], we get directly the homology of the triple loop space of $B(2n+1, 2n+2p-1)$. 

THEOREM 1.3. For an odd prime $p$, as an algebra the homologies of $\Omega^3 B(2n + 1, 2n + 2p - 1)$ are given by:

$$
\begin{align*}
H_n(\Omega^3 B(2n + 1, 2n + 2p - 1); \mathbb{F}_p) &= H_n(\Omega^3 S^{2n+1}; \mathbb{F}_p) \\
&\otimes H_n(\Omega^3 S^{2n+2p-1}; \mathbb{F}_p) \quad \text{for } n > 1,
\end{align*}
$$

$$
\begin{align*}
H_n(\Omega^3_0 B(3, 2p + 1); \mathbb{F}_p) &= \mathbb{F}_p[Q^a_{2(p-1)}(Q_{2(p-1)}[1] * [-p]) : a \geq 0] \\
&\otimes E(Q^a_{p-1}Q^b_{3(p-1)}u_{2p^2-3} : a \geq 0, b \geq 0) \\
&\otimes \mathbb{F}_p[\beta Q^a_{p-1}Q^b_{3(p-1)}u_{2p^2-3} : a \geq 0, b > 0].
\end{align*}
$$

where $\Omega^3_0 B(3, 2p + 1)$ is the zero component of $\Omega^3 B(3, 2p + 1)$.

2. Homology of the double loop space of $F_4$

From now we turn to the cases $p = 2$ and $p = 3$. The following is well-known.

THEOREM 2.1. $H^*(F_4; \mathbb{F}_2) = \mathbb{F}_2(x_3)/ (x_3^4) \otimes E(Sq^2 x_3, x_{15}, Sq^8 x_{15})$, $H^*(F_4; \mathbb{F}_3) = \mathbb{F}_3(\beta P^1 x_3)/(\beta P^1 x_3)^3 \otimes E(x_3, P^1 x_3, x_{11}, P^1 x_{11})$.

THEOREM 2.2. The cohomology of the loop space $\Omega F_4$ is

$$
H^*(\Omega F_4; \mathbb{F}_2) = \mathbb{F}_2[y_2]/(y_2^4) \otimes \Gamma(y_{8}, y_{10}, y_{14}, y_{22}).
$$

Proof. In the the Eilenberg–Moore spectral sequence converging to $H^*(\Omega F_4; \mathbb{F}_2)$,

$$
E_2 = Tor_{H^*(X; \mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2)
$$

$$
= \Gamma(\sigma(Sq^2 x_3), \sigma x_{15}, \sigma(Sq^8 x_{15})) \otimes (E(\sigma x_3)) \otimes \Gamma(\phi_2(x_3)).
$$

Since $E_2$ is even dimensional, we have $E_2 = E_\infty$. Since the Eilenberg–Moore spectral sequence is a spectral sequence of Steenrod modules, we have

$$
Sq^2(\sigma x_3) = (\sigma x_3)^2 = \sigma(Sq^2 x_3).
$$

So we get the conclusion.

THEOREM 2.3. The cohomology of the loop space $\Omega F_4$ is

$$
H^*(\Omega F_4; \mathbb{F}_3) = \mathbb{F}_3[y_2]/(y_2^9) \otimes \Gamma(y_{10}, y_{14}, y_{18}, y_{22}).
$$
Proof. In the EMSS converging to $H^*(\Omega F_4; \mathbb{Z}/(3))$

$$E_2 = Tor_{H^*(X; \mathbb{Z}/(p))}$$

$$= \Gamma(\sigma x_3, \sigma P^1 x_3, \sigma x_{11}, \sigma P^1 x_{11}) \otimes (E(\sigma \beta P^1 x_3)) \otimes \Gamma(\phi_1(\beta P^1 x_3)).$$

Since the cohomology of the loop space of a compact simple Lie group is even–dimensional and torsion free from the Morse theory, we have that $d_2(y_3(\sigma x_3)) = \sigma \beta P^1 x_3$. So

$$E_3 = \mathbb{F}_3[\sigma x_3] \otimes \Gamma(\phi_1(\beta P^1 x_3)) \otimes \Gamma(\sigma P^1 x_3, \sigma x_{11}, \sigma P^1 x_{11}).$$

Since $E_3$ is even dimensional, we have $E_3 = E_\infty$. Let $y_2 \in H^*(\Omega F_4; \mathbb{F}_3)$ represent $\sigma x_3$ and $y_{22} \in H^*(\Omega F_4; \mathbb{F}_3)$ represent $\phi_1(\beta P^1 x_3)$. Hence as a coalgebra

$$H^*(\Omega F_4; \mathbb{F}_3) = \mathbb{F}_3[y_2]/(y_2^3) \otimes \Gamma(y_{22}) \otimes \Gamma(\sigma P^1 x_3, \sigma x_{11}, \sigma P^1 x_{11}).$$

The action of the Steenrod operators on $E_\infty$ induces the algebra structure of $H^*(\Omega F_4; \mathbb{F}_3)$. In $E_\infty$, we have $(\sigma x_3)^3 = P^1 \sigma x_3 = \sigma P^1 x_3$ which in turn gives that as an algebra

$$H^*(\Omega F_4; \mathbb{F}_3) = \mathbb{F}_3[y_2]/(y_2^9) \otimes \Gamma(y_10, y_{14}, y_{18}, y_{22}).$$

As in the case of the cohomology, we can compute the Eilenberg–Moore spectral sequence converging to $H_*(\Omega F_4; \mathbb{F}_p)$ with

$$E_2 = \text{Ext}_{H^*(X; \mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p).$$

COROLLARY 2.4. As an algebra the homologies of the loop space $\Omega F_4$ are

$$H_*(\Omega F_4; \mathbb{F}_2) = E(y_2) \otimes \mathbb{F}_2[y_4, y_{10}, y_{14}, y_{22}],$$

$$H_*(\Omega F_4; \mathbb{F}_3) = \mathbb{F}_3[y_2]/(y_2^3) \otimes \mathbb{F}_3[y_6, y_{10}, y_{14}, y_{22}].$$

Now consider the path–loop fibration

$$\Omega^2 F_4 \rightarrow * \rightarrow \Omega F_4$$
THEOREM 2.5. The cohomology suspension map
\[ \sigma : Q^*(\Omega F_4; \mathbb{F}_p) \to P^{*-1}(\Omega^2 F_4; \mathbb{F}_p) \]
is a monomorphism for \( p = 2 \) or \( p = 3 \).

Proof. We divide the proof in two cases.

(Case 1) \( p = 2 \)
Assume that \( x \in (\ker \sigma)_{2q} \). From Proposition (1.2,4), \( 2q = n \times 2^i + 2 \) for some \( n \geq 1 \) and \( s \geq 2 \). Hence \( 2q \equiv 2 \) (mod 4). Let \( \tilde{F}_4 \) is the 3-connected cover of \( F_4 \). Then \( \Omega^2 F_4 \simeq S^1 \times \Omega^2 \tilde{F}_4 \). Since \( \tilde{F}_4 \) is in fact 7-connected, we have \( n \geq 6 \).

Then \( 2q \geq 6 \times 4 + 2 = 26 \) and there is no indecomposable element of dimension \( d \) with \( d \geq 26 \) and \( d \equiv 2 \) (mod 4). Hence \( \ker \sigma = 0 \) and \( \sigma \) is a monomorphism.

(Case 2) \( p = 3 \)
Since \( \Omega F_4 \) has torsion free cohomology we have \( H^{odd}(\Omega F_4; \mathbb{F}_3) = 0 \).

Assume that \( x \in (\ker \sigma)_{2q} \). From Proposition (1.2,4) we must have \( 2q = 2n3^s + 2 \) for some \( n \geq 1 \) and \( s \geq 1 \). Since \( Q_{2n}(\Omega^2 F_4; \mathbb{F}_3) \) is dual to \( P^{2n}(\Omega^2 F_4; \mathbb{F}_3) \) and \( H^{odd}(\Omega F_4; \mathbb{F}_3) = 0 \), from Proposition (1.2,3) \( P^{2n}(\Omega^2 F_4; \mathbb{F}_3) \) is obtained by the transpotence \( \phi_k \) for some \( k \). Hence \( 2n = (2m3^r - 2) \) for some \( m \geq 1 \) and \( r \geq 1 \). Then \( 2q = (2m3^r - 2)3^s + 2 \geq (2 \cdot 3 - 2)3 + 2 = 14 \) and \( 2q \equiv 2 \) (mod 3).

By inspecting \( H^*(\Omega F_4; \mathbb{F}_3) \) we can see that \( y_{14} \in H^*(\Omega F_4; \mathbb{F}_3) \) is the only possible elements in \( \ker \sigma \). But \( y_2 \) is of height greater than 3. Therefore \( y_{14} \) is not in \( \ker \sigma \). Hence \( \sigma \) is a monomorphism.

From the above theorem the Eilenberg–Moore spectral sequence converging to \( H^*(\Omega^2 F_4; \mathbb{F}_2) \) (or \( H_*(\Omega^2 F_4; \mathbb{F}_2) \)) collapses from the \( E_2 \) term and we get the following theorem.

THEOREM 2.6. As an algebra, the homology rings of the double loop space of \( F_4 \) are as follows:

(a) \( H_*(\Omega^2 F_4; \mathbb{F}_2) = E(z_1) \otimes \mathbb{F}_2[z_6] \otimes \mathbb{F}_2[Q^*_i z_i : a \geq 0, i = 7, 9, 13, 21] \),

(b) \( H_*(\Omega^2 F_4; \mathbb{F}_3) = E(z_1) \otimes \mathbb{F}_3[z_{16}] \otimes E(Q^*_i z_i : a \geq 0) \)

\( \otimes \mathbb{F}_3[\beta Q^*_i z_{17} : a > 0] \otimes E(Q^*_i z_9 : a \geq 0) \otimes \mathbb{F}_3[\beta Q^*_i z_9 : a > 0] \)

\( \otimes E(Q^*_i z_{13} : a \geq 0) \otimes \mathbb{F}_3[\beta Q^*_i z_{13} : a > 0] \otimes E(Q^*_i z_{21} : a \geq 0) \)

\( \otimes \mathbb{F}_3[\beta Q^*_i z_{21} : a > 0] \).
Proof. (a) Consider the path loop fibration: $\Omega^2 F_4 \to \ast \to \Omega F_4$. Then the elements $y_2, y_4^2, y_4^3, y_{10}, y_{14}, y_{22}$ in $H_*(\Omega^2 F_4; \mathbb{F}_2)$ are dual to generators in cohomology. Hence by Theorem 2.5, they are transgressive and let $z_1, z_7, z_9, z_{13}, z_{21}$ be the corresponding images in $H_*(\Omega^2 F_4; \mathbb{F}_2)$. Since $y_4$ is not primitive, the element $y_4$ is not transgressive and $d^2(y_4 \otimes 1) = y_2 \otimes z_1$. Since $d^2(y_4^2 \otimes 1) = 0$, $y_2 y_4 \otimes z_1$ hits $z_6 \in H_6(\Omega^2 F_4; \mathbb{F}_2)$ and $y_4^2$ transgresses to $z_7 \in H_7(\Omega^2 F_4; \mathbb{F}_2)$. By Proposition (1.1), $y_4^2 \otimes z_1$ transgresses to $Q_1^3 z_7$ and $Q_0^2 y_4 \otimes Q_1^3 z_7$ hits $(Q_1^3 z_7)^2$. For $i = 10, 14, 22$, $y_i$ transgresses to $z_{i-1} \in H_7(\Omega^2 F_4; \mathbb{F}_2)$. So we have the following:

1. $\tau(Q_0^3 y_i) = Q_1^3 z_{i-1}, a \geq 0, \ i = 10, 14, 22$
2. $d(y_2 y_4 \otimes z_1) = 1 \otimes z_6^{k+1}, k \geq 0$,
3. $\tau(y_4^{2^k+1}) = \tau(Q_0^3 y_4) = Q_1^3 z_7, a \geq 0$.

Therefore

$$H_*(\Omega^2 F_4; \mathbb{F}_2) = E(z_1) \otimes \mathbb{F}_2[z_6] \otimes \mathbb{F}_2[Q_1^a z_i : a \geq 0, i = 7, 9, 13, 21].$$

(b) The generators $y_2, y_{10}, y_{14}, y_6^2, y_{22}$ in $H_*(\Omega F_4; \mathbb{F}_3)$ are dual to the generators in $H^*(\Omega F_4; \mathbb{F}_3)$. Hence in the Serre spectral sequence corresponding to the path–loop fibration, they are transgressive by Theorem 2.5 and let $z_i$ for $i = 1, 9, 13, 21$ be the corresponding trangressive images in $H_*(\Omega^2 F_4; \mathbb{F}_3)$. Since $y_6$ is not primitive, the element $y_6$ can not be transgressive and $d^2(y_6 \otimes 1) = y_2 \otimes z_1$. We also have $d^2(y_6 \otimes 1) = 0$ and $E_3^{r,s} = 0$ for $1 \leq r \leq 15, r \neq 10, 14$ and $2 \leq s \leq 9$. $y_6^2 \otimes z_1$ hits $z_{16} \in H_{16}(\Omega^2 F_4; \mathbb{F}_3)$. Therefore we have the following in the Serre spectral sequence corresponding the path–loop fibration:

1. $\tau(Q_0^3 y_i) = Q_1^3 z_{i-1}, a \geq 0, \ i = 10, 14, 22$
2. $d((Q_0^3 y_i)^2 \otimes (Q_2^3 z_{i-1})(\beta Q_2^{a+1} z_{i-1})) = 1 \otimes (\beta Q_2^{a+1} z_{i-1})^{k+1}, a \geq 0, k \geq 0$,
3. $d((y_6 z_6 \otimes z_1) = 1 \otimes z_{16}^{k+1}, k \geq 0$,
4. $\tau(y_6^{3^k+1}) = \tau(Q_0^3 y_6) = Q_1^3 z_{17}, a \geq 0$,
5. $d((y_6)^{3^k+1} \otimes (Q_2^3 z_{17})(\beta Q_2^{a+1} z_{17})) = 1 \otimes (\beta Q_2^{a+1} z_{17})^{k+1}, a \geq 0, k \geq 0$.

Therefore the conclusion follows.

Remark. In fact we can get the relations such that $\beta z_7 = z_6$ and $\beta z_{17} = z_{16}$ from the homology of the double loop space of the 3-connected cover $\tilde{F}_4$ [13].
3. Homology of the triple loop space of $F_4/\text{Spin}(9)$ and $F_4$

We have the following fibration

$$\text{Spin}(9) \xrightarrow{i} F_4 \xrightarrow{p} F_4/\text{Spin}(9)$$

where $F_4/\text{Spin}(9) = S^8 \cup e^{16}$ is the Cayley projective plane such that

$$H^*(F_4/\text{Spin}(9); \mathbb{F}_2) = \mathbb{F}_2[x_8]/(x_8^3).$$

So we have a sequence of fibrations:

$$\cdots \to \Omega^n \text{Spin}(9) \xrightarrow{\Omega^n i} \Omega^n F_4 \xrightarrow{\Omega^n p} \Omega^n F_4/\text{Spin}(9) \to \cdots$$

$$\cdots \to \Omega F_4/\text{Spin}(9) \xrightarrow{\gamma} \text{Spin}(9) \xrightarrow{i} F_4 \xrightarrow{p} F_4/\text{Spin}(9)$$

In this section we first study the homology of the triple loop space of the Cayley projective plane and study $H_*(\Omega^3 F_4; \mathbb{F}_p)$ later. We do the case of $p = 2$ first.

**Lemma 3.1.** As an algebra,

$$H_*(\Omega^3 F_4/\text{Spin}(9); \mathbb{F}_2) = \mathbb{F}_2[Q^a_1u_5 : a \geq 0] \otimes \mathbb{F}_2[Q^a_1Q^b_2u_20 : a \geq 0, b \geq 0].$$

**Proof.** In the Eilenberg–Moore spectral sequence converging to $H^*(\Omega F_4/\text{Spin}(9); \mathbb{F}_2)$, we have

$$E_2 = \text{Tor}_{H^*(F_4/\text{Spin}(9); \mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2) = E(y_7) \otimes \Gamma(y_{22}).$$

By the degree reason, there can not be non–trivial differential. Hence $E_2 = E_\infty$ and we get

$$H^*(\Omega F_4/\text{Spin}(9); \mathbb{F}_2) = E(y_7) \otimes \Gamma(y_{22}).$$

Consider the Eilenberg–Moore spectral sequence converging to $H_*(\Omega^2 F_4/\text{Spin}(9); \mathbb{F}_2)$ with

$$E_2 = \text{Ext}^{H^*(\Omega F_4/\text{Spin}(9); \mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[z_6] \otimes \mathbb{F}_2[Q^a_1z_{21} : a \geq 0].$$
This is a spectral sequence of a Hopf algebra. So the source of the first non-trivial differential is an indecomposable element and the target of the first non-trivial differential is a primitive element. In $E_2$, all generators are of bidegree $(-1, 7)$ or of bidegree $(-1, 22 \times 2^n)$ for $n \geq 0$. So the bidegree of the target of the first differential is of the form $(-1 - m, 6 + m)$ or $(-1 - m, 22 \times 2^n + m - 1)$. In the other hand every primitive element in the $E_2$-term is of bidegree $(-2m, 7 \times 2^n)$ or $(-2m, 22 \times 2^{m+n})$ for any $m, n \geq 0$. But $22 \times 2^n - 2 = 2(11 \times 2^n - 1)$ can not be of the form $7 \times 2^m - 2^m = 2(2^m \times 3)$ for any $m, n \geq 0$. Therefore there is no non trivial differential. Hence the spectral sequence collapses from the $E^2$-term and we have

$$H_*(\Omega^2 F_4/\text{Spin}(9); \mathbb{F}_2) = \mathbb{F}_2[z_6] \otimes \mathbb{F}_2[Q_i^a z_{21} : a \geq 0].$$

Consider the following fibration

$$\Omega^3 F_4/\text{Spin}(9) \longrightarrow \Omega^2 \text{Spin}(9) \longrightarrow \Omega^2 F_4$$

and the corresponding Serre spectral sequence with

$$E^2 = H_*(\Omega^2 F_4; \mathbb{F}_2) \otimes H_*(\Omega^3 F_4/\text{Spin}(9); \mathbb{F}_2).$$

From [5] we know that

$$H_*(\Omega^2 \text{Spin}(9); \mathbb{F}_2) = E(z_1) \otimes \mathbb{F}_2[z_6] \otimes \mathbb{F}_2[Q_i^a z_i : a \geq 0, i = 5, 7, 9, 13].$$

From Theorem 2.6,

$$H_*(\Omega^2 F_4; \mathbb{F}_2) = E(z_1) \otimes \mathbb{F}_2[z_6] \otimes \mathbb{F}_2[Q_i^a z_i : a \geq 0, i = 7, 9, 13, 21].$$

Since the homology of the total space has a 5-dimensional generator and has no 21-dimensional generator, the homology of the fibre space must have a 5-dimensional generator, say $u_5$, and a 20-dimensional element, say $u_20$, which should be the target of the differential from $z_{21}$. Since the homology of the total space contains $Q^a_1 z_5 \ a \geq 0$, by the naturality of the action of the Dyer- Lashof operators, the homology of the fibre space also has $Q^a_1 u_5$. Now we consider the Eilenberg–Moore spectral sequence converging to $H_*(\Omega^3 F_4/\text{Spin}(9); \mathbb{F}_2))$ with

$$E^2 = \text{Cotor}_{H_*}^{\Omega^2 F_4/\text{Spin}(9); \mathbb{F}_2}^{\mathbb{F}_2} = \mathbb{F}_2[Q^a_1 u_5 : a \geq 0] \otimes \mathbb{F}_2[Q^a_1 Q^b_2 u_20 : a, b \geq 0].$$

Then the spectral sequence collapses from $E^2$ and we get the conclusion.

Since $\pi_3(F_4) = \mathbb{Z}, \pi_0(\Omega^3 F_4) = \mathbb{Z}$. Let $\Omega^3_0 F_4$ be the zero component of $\Omega^3 F_4$.
THEOREM 3.2. As an algebra,
\[ H_*(\Omega^3 Spin(9); \mathbb{F}_2) = \mathbb{F}_2[Q^a_1 u_5 : a \geq 0] \]
\[ \otimes \mathbb{F}_2[Q^a_1 Q^b_2 u_i : i = 6, 8, 12, 20, a \geq 0, b \geq 0]. \]

Proof. Consider the following sequence of fibrations:
\[ \Omega^3 Spin(9) \to \Omega^3 F_4 \to \Omega^3 F_4/Spin(9) \to \Omega^2 Spin(9) \]
and the Serre spectral sequence converging to \( H_*(\Omega^3 F_4; \mathbb{F}_2) \) with
\[ E^2 = H_*(\Omega^3 F_4/Spin(9); \mathbb{F}_2) \otimes H_*(\Omega^3 Spin(9); \mathbb{F}_2). \]

From [5] we know that
\[ H_*(\Omega^3 Spin(9); \mathbb{F}_2) = \mathbb{F}_2[Q^a_1 u_5 : a \geq 0] \]
\[ \otimes \mathbb{F}_2[Q^a_1 Q^b_2 u_i : i = 4, 6, 8, 12, a \geq 0, b \geq 0]. \]

From the proof of Lemma 3.1 we know that
\[ (\Omega^2 \gamma)_*(Q^a_1(u_5)) = Q^a_1(z_5), a \geq 0. \]
Since \( \Omega^2 \gamma \circ \Omega^3 p \) is null-homotopic, \( Q^a_1(u_5) \) can not survive to \( E^\infty \). This means that we have the following differentials
\[ d(Q^a_0 Q^b_1(u_5)) = Q^a_1 Q^b_2(u_4), \quad a \geq 0, b \geq 0 \]
from the properties of the Dyer-Lashof operations in Proposition (1.1, (1)).
Note that \( u_4 \) is the unique 4-dimensional primitive element in \( H_*(\Omega^3 Spin(9); \mathbb{F}_2) \).
Since this is a spectral sequence of a Hopf algebra, we have
\[ \Delta_*(d(u_{20})) = d(\Delta_*(u_{20})). \]
Hence \( d(u_{20}) \) should be primitive since \( u_{20} \) is primitive. But there is no other 19-dimensional primitive element than \( Q^a_1 u_4 \) in \( H_*(\Omega^3 Spin(9); \mathbb{F}_2) \), so that \( d(u_{20}) = 0 \) up to choices of generators. Hence by the properties of the Dyer-Lashof operations, we have
\[ d(Q^a_1 Q^b_2(u_{20})) = 0, \]
so that \( Q^a_1 Q^b_2(u_{20}) \) survives permanently for each \( a, b \geq 0 \). Hence the result follows.

Now we turn to the case of the homology with \( \mathbb{F}_3 \) coefficients.
LEMMA 3.3. [4,12] As an algebra,

\[ H_*(\Omega^2 SU(9); \mathbb{F}_3) = E(Q^a_2 z_{2i-1} : i = 1, 2, 4, 5, 7, 8, a \geq 0) \]
\[ \otimes \mathbb{F}_3[Q^a_4 z_i : i = 16, 34, a \geq 0] \otimes \mathbb{F}_3[\beta Q^a_2 z_{2i-1} : i = 4, 5, 7, 8, a > 0]. \]

\[ H_*(\Omega^3 SU(9); \mathbb{F}_3) = \mathbb{F}_3[Q^a_4 (Q^a_4 [1] * [-3]) : a \geq 0] \]
\[ \otimes \mathbb{F}_3[Q^a_4 u_{2i-2} : i = 2, 4, 5, 7, 8, a \geq 0] \]
\[ \otimes E(Q^a_2 Q^b_6 u_{6i-3} : i = 3, 6, a \geq 0, b \geq 0) \]
\[ \otimes \mathbb{F}_3[\beta Q^a_2 Q^b_6 u_{6i-3} : i = 3, 6, a > 0, b \geq 0] \]
\[ \otimes E(Q^a_2 \beta Q^b_6 u_{2i-2} : i = 4, 5, 7, 8, a \geq 0, b > 0) \]
\[ \otimes \mathbb{F}_3[\beta Q^a_2 \beta Q^b_6 u_{2i-2} : i = 4, 5, 7, 8, a > 0, b > 0], \]

where \([1] \in H_*(\Omega^3 SU(9); \mathbb{F}_3)\) is the image of the generator in \(\tilde{H}_0(S^0)\) for the map: \(S^0 \to \Omega^3 SU(9)\).

By the Harris splitting [9],

\[ SU(2n + 1) \simeq (SU(2n+1)/SO(2n+1)) \times SO(2n+1). \]

Hence from the cases of the homologies of the double and triple loop spaces of \(SU(9)[12]\), we easily get the following corollary.

COROLLARY 3.4. As an algebra,

\[ H_*(\Omega^2 Spin(9); \mathbb{F}_3) = H_*(\Omega^2 SO(9); \mathbb{F}_3) \]
\[ = E(Q^a_2 z_i : a \geq 0) \otimes \mathbb{F}_3[Q^a_4 z_{16} : a \geq 0] \otimes E(Q^a_2 z_9 : a \geq 0) \]
\[ \otimes \mathbb{F}_3[\beta Q^a_2 z_9 : a > 0] \otimes E(Q^a_2 z_{13} : a \geq 0) \otimes \mathbb{F}_3[\beta Q^a_2 z_{13} : a > 0], \]

\[ H_*(\Omega^3 Spin(9); \mathbb{F}_3) = H_*(\Omega^3 SO(9); \mathbb{F}_3) \]
\[ = \mathbb{F}_3[Q^a_4 (Q^a_4 [1] * [-3]) : a \geq 0] \otimes \mathbb{F}_3[Q^a_4 u_i : 8, 12, a \geq 0] \]
\[ \otimes E(Q^a_2 Q^b_6 u_{15} : a \geq 0, b \geq 0) \otimes \mathbb{F}_3[\beta Q^a_2 Q^b_6 u_{15} : a > 0, b \geq 0] \]
\[ \otimes E(Q^a_2 \beta Q^b_6 u_i : i = 8, 12, a \geq 0, b > 0) \]
\[ \otimes \mathbb{F}_3[\beta Q^a_2 \beta Q^b_6 u_i : i = 8, 12, a > 0, b > 0]. \]

Similar to the case of the coefficients \(\mathbb{F}_2\), by computing Eilenberg–Moore spectral sequence twice we get the following lemma.

LEMMA 3.5. As an algebra,

\[ H_*(\Omega^2 F_4/Spin(9); \mathbb{F}_3) = \mathbb{F}_3[z_6] \otimes E(Q^a_2 z_{2i} : a \geq 0) \]
\[ \otimes \mathbb{F}_3[\beta Q^a_2 z_{2i} : a > 0]. \]
From Theorem 2.6, we can analyze the Serre spectral sequence converging to $H_*(\Omega^2 F_4; \mathbb{F}_3)$ for the following fibration

$$\Omega^2 Spin(9) \xrightarrow{\Omega^2 i} \Omega^2 F_4 \xrightarrow{\Omega^2 p} \Omega^2 F_4/Spin(9).$$

Then we have the following differentials

$$d(Q^a_0 u_6) = Q^{a+1}_2 z_1, \quad a \geq 0.$$ 

And $(Q^a_0 u_6)^2 Q^{a+1}_2 z_1$ survives eventually and becomes $E(Q^a_2 z_{17})$ in $H_*(\Omega^2 F_4; \mathbb{F}_3)$. Then we have

$$\begin{align*}
\Omega^2 i_*(Q^a_2 z_j) &= Q^a_2 z_j, \quad a \geq 0, \quad j = 9, 13, 21 \\
\Omega^2 i_*(\beta Q^a_2 z_j)^k &= (\beta Q^a_2 z_j)^k, \quad a > 0, \quad j = 9, 13, 21, k \geq 0, \\
\Omega^2 i_*(u_{16}) &= u_{16}, \\
\Omega^2 i_*(Q^a_2 u_{16})^k &= (\beta Q^a_2 z_{17})^k, \quad a > 0, k \geq 0.
\end{align*}$$

**Lemma 3.6.** As an algebra,

$$H_*(\Omega^3 F_4/Spin 9; \mathbb{F}_3) = E(Q^a_2 u_5 : a \geq 0) \otimes_\mathbb{F}_3 [Q^a_4 u_{16} : a \geq 0] \otimes \mathbb{F}_3 [Q^a_4 u_{20} : a \geq 0] \otimes E(Q^a_2 \beta Q^b_4 u_{20} : a \geq 0, b > 0) \otimes \mathbb{F}_3 [\beta Q^a_2 \beta Q^b_4 u_{20} : a > 0, b > 0].$$

**Proof.** Consider the following morphisms of fibrations:

$$\begin{align*}
\Omega^3 F_4 & \xrightarrow{\ast} \Omega^2 F_4 \\
\downarrow & \quad \downarrow \\
\Omega^3 F_4/Spin(9) & \xrightarrow{\Omega^2 \gamma} \Omega^2 Spin(9) \xrightarrow{\Omega^2 i} \Omega^2 F_4
\end{align*}$$

Now we study the Serre spectral sequence converging to $H_*(\Omega^2 Spin(9); \mathbb{F}_3)$ with

$$E^2 = H_*(\Omega^2 F_4; \mathbb{F}_3) \otimes H_*(\Omega^3 F_4/Spin(9); \mathbb{F}_3).$$

$H_*(\Omega^2 Spin(9); \mathbb{F}_3)$ has the generator $Q_2 z_1$ but $H_*(\Omega^2 F_4; \mathbb{F}_3)$ has no generator of degree 5. So the homology of the fiber space must contain the element of degree 5, say $u_5$, and by the naturality of the action of the
Homology of the triple loop space of the exceptional Lie group $F_4$

Dyer–Lashof operators the homology of the fiber space contains $Q_2^a u_5$ for $a \geq 0$. In $H_*(\Omega^2 Spin(9); \mathbb{F}_3)$ there are no generators of the same degree as $Q_2^a z_{17}$, $a \geq 0$. So by naturality the homology of the path–loop fibration by Proposition 1.2. So by naturality the homology of $\Omega^3 F_4 / Spin(9)$ have $Q_2^a u_16$, $a > 0$ such that

$$d(Q_2^a z_{17}) = Q_4^a u_16, a \geq 0.$$ 

From the facts

$$\Omega^2 p_*(Q_2^a z_{21}) = Q_4^a z_{21}, a \geq 0,$$
$$\Omega^2 p_*((\beta Q_2^a z_{21})^k) = (\beta Q_4^a z_{21})^k, a > 0, k \geq 0,$$

we have differentials from $Q_2^a z_{21}, a \geq 0$ and $Q_0^a \beta Q_2^b z_{21}, a \geq 0, b > 0$ such that

$$d(Q_2^a z_{21}) = Q_4^a u_{20},$$
$$d(Q_0^a \beta Q_2^b z_{21}) = Q_2^a \beta Q_4^b u_{20},$$
$$d((Q_0^a \beta Q_2^b z_{21}))^2 Q_2^a \beta Q_4^b u_{20}) = -\beta Q_2^{a+1} \beta Q_4^b u_{20}.$$

Now we consider the Eilenberg–Moore spectral sequence for the above fibration converging to $H_*(\Omega^3 F_4 / Spin(9); \mathbb{F}_3)$ with

$$E^2 = \text{Cotor}^{H_*(\Omega^2 F_4; \mathbb{F}_3)}(\mathbb{F}_3, H_*(\Omega^2 Spin(9); \mathbb{F}_3))$$
$$= \text{Cotor}^{H_*(\Omega^2 F_4; \mathbb{F}_3) / \Omega^2 i_*}(\mathbb{F}_3, \mathbb{F}_3) \otimes E(Q_2^a u_5 : a \geq 0)$$
$$= E(Q_2^a u_5 : a \geq 0) \otimes \mathbb{F}_3 [Q_4^a u_{16} : a \geq 0] \otimes \mathbb{F}_3 [Q_4^a u_{20} : a \geq 0]$$
$$\otimes E(Q_2^a \beta Q_4^b u_{20} : a \geq 0, b > 0) \otimes \mathbb{F}_3 [\beta Q_2^a \beta Q_4^b u_{20} : a > 0, b > 0].$$

The above information about the Serre spectral sequence implies that the Eilenberg–Moore spectral sequence collapses from $E^2$ and we get the conclusion.

**Theorem 3.7.** As an algebra,

$$H_*(\Omega^3 F_4; \mathbb{F}_3) = \mathbb{F}_3 [Q_4^a u_{16} : a \geq 0] \otimes E(Q_2^a Q_4^b u_{15} : a \geq 0, b \geq 0)$$
$$\otimes \mathbb{F}_3 [\beta Q_2^a Q_4^b u_{15} : a > 0, b \geq 0] \otimes E(Q_2^a \beta Q_4^b u_i : i = 8, 12, 20, a \geq 0, b > 0)$$
$$\otimes \mathbb{F}_3 [Q_4^a u_i : i = 8, 12, 20, a \geq 0] \otimes \mathbb{F}_3 [\beta Q_2^a \beta Q_4^b u_i : i = 8, 12, 20, a > 0, b > 0].$$
Proof. We have the following sequence of fibrations

\[ \Omega^3 \text{Spin}(9) \longrightarrow \Omega^3 F_4 \longrightarrow \Omega^3 F_4/\text{Spin}(9) \xrightarrow{\Omega^2\gamma} \Omega^2 \text{Spin}(9). \]

Consider the Serre spectral sequence converging to \( H_*(\Omega^3 F_4/\text{Spin}(9); \mathbb{F}_3) \) with

\[ E^2 = H_*(\Omega^3 F_4/\text{Spin}(9); \mathbb{F}_3) \otimes H_*(\Omega^3 \text{Spin}(9); \mathbb{F}_3). \]

By Corollary 3.4 we have

\[ H_* (\Omega^3 \text{Spin}(9); \mathbb{F}_3) \]

\[ = \mathbb{F}_3[Q_4^a (Q_4[1] * [-3]) : a \geq 0] \otimes \mathbb{F}_3[Q_4^a u_i : 8, 12, a \geq 0] \]

\[ \otimes E(\beta Q_2^a Q_0^b u_{15} : a \geq 0, b \geq 0) \otimes \mathbb{F}_3[\beta Q_2^a Q_0^b u_{15} : a > 0, b \geq 0] \]

\[ \otimes \mathbb{F}_3[\beta Q_2^a \beta Q_0^b u_i : i = 8, 12, a \geq 0, b > 0] \]

\[ \otimes \mathbb{F}_3[\beta Q_2^a \beta Q_0^b u_i : i = 8, 12, a > 0, b > 0]. \]

From Lemma 3.6, we have

\[ \Omega^2 \gamma_* (Q_2^a u_5) = Q_2^{a+1} z_1, a \geq 0. \]

So from the uniqueness of the 4-dimensional primitive element we have the following differential

\[ d(Q_2^a u_5) = Q_2^a (Q_4[1] * [-3]). \]

Since there is no 19 dimensional primitive element in \( H_*(\Omega^3 \text{Spin}(9); \mathbb{F}_3) \), the element \( u_{20} \) survives. Now we should determine whether there is a differential from \( u_{16} \) to the 15-dimensional primitive element \( u_{15} \). Consider the Eilenberg–Moore spectral sequence converging to \( H_*(\Omega^3 F_4; \mathbb{F}_3) \) with

\[ E^2 = \text{Cotor}^{H_*(\Omega^3 F_4; \mathbb{F}_3)}(\mathbb{F}_3, \mathbb{F}_3). \]

Then the 16-dimensional indecomposable element lies in (-1, 17) and the 15-dimensional primitive element lies in (-1, 16). Therefore there can not be non trivial differential. Hence the differential from \( x_{16} \) is trivial. So we get the conclusion.

For each odd prime \( p \), there exist a simply connected finite CW complex \( X \) whose localization \( X(p) \) at \( p \) is an \( H \)-space with

\[ H^*(X(p); \mathbb{F}_3) = \mathbb{F}_3[\beta \mathcal{P}^1 x_3]/((\beta \mathcal{P}^1 x_3)^3) \otimes E(x_3, \mathcal{P}^1 x_3). \]

Though \( F_4 \) is not quasi 3-regular, it can be decomposed as a product [8];

\[ F_4 \cong (3) X(3) \times B(11, 15). \]

So from the homology of triple loop space of \( F_4 \), we get that
COROLLARY 3.8. As an algebra,

\[
H_*(\Omega^3 X(3); \mathbb{F}_3) = \mathbb{F}_3[Q_4^a u_{16} : a \geq 0] \\
\otimes E(Q_2^a Q_6^b u_{15} : a \geq 0, b \geq 0) \\
\otimes \mathbb{F}_3[\beta Q_2^a Q_4^b u_{15} : a > 0, b \geq 0] \\
\otimes E(Q_2^a \beta Q_4^b u_{20} : a \geq 0, b \geq 0) \\
\otimes \mathbb{F}_3[\beta Q_4^a u_{20} : a \geq 0] \otimes \mathbb{F}_3[\beta Q_2^a \beta Q_4^b u_{20} : a > 0, b > 0].
\]

References


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