A REMARK ON HALF-FACTORIAL DOMAINS

Heung-Joon Oh

Abstract. An atomic integral domain $R$ is a half-factorial domain (HFD) if whenever $x_1 \cdots x_m = y_1 \cdots y_n$ with each $x_i, y_j \in R$ irreducible, then $m = n$. In this paper, we show that if $R[X]$ is an HFD, then $\text{Cl}(R) \cong \text{Cl}(R[X])$, and if $G_1$ are torsion abelian groups, then there exists a Dedekind HFD $R$ such that $\text{Cl}(R) = G_1 \oplus G_2$.

1. Introduction

Let $R$ be an integral domain and $R^* = R - \{0\}$. A nonunit $r \in R^*$ is said to be irreducible if whenever $r = ab, a, b \in R$, either $a$ or $b$ is a unit of $R$. An integral domain $R$ is atomic if every nonzero nonunit of $R$ can be factored as a product of irreducible elements of $R$. Following Zaks [13], we say that an atomic integral domain $R$ is a half-factorial domain (HFD) if whenever $x_1 \cdots x_m = y_1 \cdots y_n$ with each $x_i, y_j \in R$ irreducible, then $m = n$. It is well known that any Krull domain $R$ with divisor class group $\text{Cl}(R) = \mathbb{Z}_2$ is a HFD, but not a UFD. It is classical that a ring of integers $R$ of a number field is a UFD if and only if $\text{Cl}(R) = \{0\}$. The first arithmetic description of rings of integers with nontrivial divisor class groups was given in Carlitz [8]. He proved that $|\text{Cl}(R)| \leq 2$ if and only if any two factorizations of an element of $R$ into irreducible elements have the same number of factors. Thus a Dedekind domain $R$ with the property that each nonzero ideal class contains a prime ideal is a HFD if and only if $|\text{Cl}(R)| \leq 2$. The ring of integers in a finite algebraic number field over the rational is an example of a Dedekind domain which satisfies the condition of having a prime ideal in each ideal class. In order to measure how far an atomic integral domain $R$ is from being a HFD, we define the elasticity of $R$ as

$$\rho(R) = \sup \{m/n | x_1 \cdots x_m = y_1 \cdots y_n, \text{ for } x_i, y_j \in R \text{ irreducible}\}.$$ 

Thus $1 \leq \rho(R) \leq \infty$ and $\rho(R) = 1$ if and only if $R$ is a HFD. This concept was introduced by Valenza [12], who studied $\rho(R)$ for $R$ the ring of integers in
an algebraic number field. In this paper, we show that if \( R[X] \) is an HFD, then \( \text{Cl}_t(R) \cong \text{Cl}_t(R[X]) \), and if \( G_1 \) and \( G_2 \) are torsion abelian groups, then there exists a Dedekind HFD \( R \) such that \( \text{Cl}(R) = G_1 \oplus G_2 \).


2. Half-Factorial and Locally Half-Factorial Domains

Let \( R \) be an integral domain. We say that \( R \) is a \emph{GCD-domain} if any two elements of \( R \) have a GCD in \( R \). In Anderson, Chapman & Smith [4], an integral domain \( R \) is said to be a \emph{locally half-factorial domain} (LHFD) if each localization \( R_S \) of \( R \) os a HFD.

Given a Dedekind domain \( R \), let \( \text{Cl}(R) \) denote its divisor class group, and \([I]\) the ideal class of \( I \) in \( \text{Cl}(R) \). If for a given abelian group \( G \) and subset \( C \subseteq G - \{0\} \) there exists a Dedekind domain \( R \) such that \( \text{Cl}(R) \cong G \) and \( C = \{c|c \in G \text{ and } c \text{ contains a non-principal prime ideal of } R\} \), then the pair \( \{G, C\} \) is called \emph{realizable} Grams [11].

Let \( G \) be an abelian group and \( C \subseteq G \). We say that \( C \) is an independent set in \( G \) if \( n_1c_1 + \cdots + n_kc_k = 0, \ n_k \in \mathbb{Z}, \ \text{distinct } c_i \in C, \) implies that each \( n_i c_i = 0 \). In fact, if \( R \) is a Dedekind domain with the torsion divisor class group \( \{\text{Cl}(R), A\} \) with \( A \) independent, then \( R \) is an LHFD [4, Theorem 2.5].

We start with following example:

**Example 2.1.** (1) Let \( R \) be a Dedekind domain with realizable pair \( \{\mathbb{Z}, A\} \) where \( A = \{-2,1,2,3,\cdots\} \) [11, Theorem 2.4]. Then \( R \) is a HFD [7, Corollary 3.3]. Thus there exists a Dedekind HFD \( R' \) such that \( \{\mathbb{Z} \oplus \mathbb{Z}, A \oplus A\} \) is a realizable pair, where \[
A \oplus A = \{(−2,0), (1,0), (2,0), \ldots (0,−2), (0,1), (0,2), \ldots\}
\]
[6, Theorem 3.1]. Now, for each nonzero nonunit \( f \in R' \), we have that \( \text{Cl}(R'_f) \) is one of the followings:

\[
\mathbb{Z} \oplus \mathbb{Z}, \ \mathbb{Z}_2 \oplus \mathbb{Z}, \ \{0\} \oplus \mathbb{Z}, \ \mathbb{Z} \oplus \{0\}, \ \mathbb{Z} \oplus \mathbb{Z}_2, \ \mathbb{Z}_2 \oplus \mathbb{Z}_2, \ \{0\}.
\]

Suppose now that \( \text{Cl}(R'_f) = \mathbb{Z}_2 \oplus \mathbb{Z} \). Then the prime ideals of \( R'_f \) are distributed in the classes \( \{(1,0), (0,−2), (0,1), (0,2), \ldots\} \). If \( \text{Cl}(R'_f) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), then the prime ideals of \( R'_f \) are distributed in the classes \( \{(1,0), (0,1)\} \). Since it is independent, \( R'_f \) is a HFD, and hence \( R' \) is a LHFD.
Example 2.2. Let $R$ be a Dedekind domain with realizable pair $\{\bigoplus_{i=1}^{\infty} \mathbb{Z}_{n_i}, A\}$, where $A = \{(1,0,0,\ldots),(0,1,0,\ldots),\ldots\}$. Then $A$ is independent, and hence $R$ is a HFD and LHFD. On the other hand, if $R$ is a Dedekind domain associated to the realizable pair $\{\mathbb{Z}_0, \{1,3\}\}$, then $R$ is a HFD and LHFD. But $\{1,3\}$ is not an independent set.

A saturated multiplicative set $S$ of $R$ is called a splitting multiplicative set Anderson, Anderson & Zafrullah [2] if for each nonzero $d \in R$, $d = sa$ for some $s \in S$ and $a \in R$ with $s'R \cap aR = s'aR$ for all $s' \in R$. The set $T = \{0 \neq t|sR \cap tR = stR \text{ for all } s \in S\}$ is also a splitting set and we call $T$ the complementary multiplicative set for $S$. A splitting multiplicative set $S$ of $R$ is said to be an lcm-splitting set if for each $s \in S$ and $d \in R$, $sR \cap dR$ is principal. $S$ is a splitting set of $R$ if and only if $R_T$ is a GCD-domain, where $T$ is the complementary multiplicative set [2, Proposition 2.4].

For an integral domain $R$, let $Cl_t(R)$ denote the $t$-class group of $R$, i.e., the group of $t$-invertible $t$-ideals of $R$ modulo its subgroup of principal fractional ideals.

For example, if $R$ is an integral domain such that each nonunit element of $R$ is a product of primary element, then $Cl_t(R) = \{0\}$. In particular, if $R$ is atomic, then $\rho(R) = \sup\{\rho(R_P)|htP = 1\}$ [6, Corollary 2.5].

Theorem 2.3. Let $R$ be an atomic domain and let $S$ be an lcm-splitting multiplicative set. Then

1. $\rho(R) = \rho(R_S)$ and $Cl_t(R) \cong Cl_t(R_S)$.

2. If $R[X]$ is a HFD, then $Cl_t(R) \cong Cl_t(R[X])$.

Proof. (1) Let $T$ be the complementary multiplicative set of $S$. Then $\rho(R) = \max\{\rho(R_S), \rho(R_T)\}$ [6, Theorem 2.3]. By [2, Proposition 2.4], $R_T$ is a GCD-domain; so $R_T$ is a UFD. Thus $\rho(R_T) = 1$ and hence $\rho(R) = \rho(R_S)$. By [2, Theorem 4.1], $Cl_t(R) \cong Cl_t(R_S)$. (2) suppose now that $R[X]$ is a HFD. Then $R$ is integrally closed [9, Theorem 2.2], and hence $Cl_t(R) \cong Cl_t(R[X])$ [10, Theorem 3.6].

With the notation in Theorem 2.3, $S$ is an lcm-splitting multiplicative set if and only if $S$ is generated by principal primes [2, Corollary 2.7], [6, Theorem 1.6]. In particular, $R$ is a HFD if and only if $R_S$ is a HFD.

Theorem 2.4. Let $G_1$ and $G_2$ be torsion abelian groups. Then there exists a Dedekind HFD $R$ such that $Cl_t(R) = G_1 \oplus G_2$.

Proof. Let $R_1, R_2$ be a Dedekind HFDs with $Cl(R_i) = G_i[1, \text{Theorem 3.2}$. Suppose that $R_i$ is associated to $\{G_i, A_i\}$ with $i = 1, 2$. Define $A = \{(1,0), (0,b)|a \in A_1, b \in$
Then $A$ is realizable Grams [11]. Let $R$ be a Dedekind domain associated to $\{G, A\}$. Then $\rho(R) = \max\{\rho(R_1), \rho(R_2)\} = 1[6, \text{Theorem 3.1}].$ Thus $R$ is a HFD.

In view of the above theorem, we have:

**Corollary 2.5.** Let $\{R_i\}$ be a finite family of Dedekind HFDs with torsion divisor class groups $\{Cl(R_i) = G_i\}$. Then there exists a Dedekind HFD $R$ such that $Cl(R) \cong \bigoplus G_i$.

**References**


Division of General Education, Cho-Dang University, 419, Seongnam-ri, Muan-eub, Muan-gn, Jeonnam 534-701, Korea