

ON SOME PROPERTIES OF BOUNDED X^* -VALUED FUNCTIONS

BOK DONG YOO

ABSTRACT. Suppose that X is a Banach space with continuous dual X^{**} , (Ω, Σ, μ) is a finite measure space. $f : \Omega \rightarrow X^*$ is a weakly measurable function such that $x^{**}f \in L_1(\mu)$ for each $x^{**} \in X^{**}$ and $T_f : X^{**} \rightarrow L_1(\mu)$ is the operator defined by $T_f(x^{**}) = x^{**}f$. In this paper we study the properties of bounded X^* -valued weakly measurable functions and bounded X^* -valued *weak** measurable functions.

1. INTRODUCTION

Suppose that X is a Banach space with continuous dual X^{**} , (Ω, Σ, μ) is a finite measure space. $f : \Omega \rightarrow X^*$ is a weakly measurable function such that $x^{**}f \in L_1(\mu)$ for each $x^{**} \in X^{**}$ and $T_f : X^{**} \rightarrow L_1(\mu)$ is the operator defined by $T_f(x^{**}) = x^{**}f$.

In this paper we study the properties of bounded X^* -valued weakly measurable functions and bounded X^* -valued *weak** measurable functions.

Throughout the paper B_X will denote the unit ball of X . An operator $T_f : X^{**} \rightarrow L_1(\mu)$ is said to be $(w^*$, norm)-continuous provided that net $T_f(x_j^{**})$ converges to $T_f(x^{**})$ in the norm topology of $L_1(\mu)$ whenever (x_j^{**}) is a net which converges to x^{**} in the *weak** topology of X^{**} .

A function $f : (\Omega, \Sigma, \mu) \rightarrow X^*$ is weakly measurable if $x^{**}f$ is measurable for every $x^{**} \in X^{**}$. A function $f : (\Omega, \Sigma, \mu) \rightarrow X^*$ is *weak** measurable if $x f$ is measurable for every $x \in X$.

An operator $T_f : X^{**} \rightarrow L_1(\mu)$ which is defined by $T_f(x^{**}) = x^{**}f$ is weakly compact if the norm closure of $T_f(B_{X^{**}})$ is weakly compact. A subset K of $L_1(\mu)$ is called uniformly integrable if $\lim_{\mu(E) \rightarrow 0} \int_E |f| d\mu = 0$ uniformly in $f \in K$.

1991 *Mathematics Subject Classification*. Primary 46B20.

2. MAIN THEOREMS

Theorem 1. *If $f : \Omega \rightarrow X^*$ is bounded weakly measurable function, then f is (w^*, norm) -sequentially continuous.*

Proof. If x_n^{**} converges to x^{**} in the weak* topology of X^{**} , then $x_n^{**}f$ converges to $x^{**}f$ pointwise. Since x_n^{**} converges to x^* in the weak* topology of X^{**} , by the principle of uniform boundedness $\sup_{n \rightarrow \infty} \|x_n^{**}\| < \infty$ and by hypothesis there exists $M > 0$ such that $\sup_{x \in \Omega} \|f\| < M$.

Since $|x_i^{**}| \leq \sup \|x_n^{**}\| \leq M$, by Lebesgue's bounded convergence theorem

$$\lim_{n \rightarrow \infty} \|x_n^{**}f - x^{**}f\| = \lim_{n \rightarrow \infty} \int_{\Omega} |x_n^{**}f - x^{**}f| d\mu = 0$$

Thus T_f is (w^*, norm) -sequentially continuous. \square

Lemma. *A subset of $L_1(\mu)$ be relatively weakly compact if and only if it is bounded and uniformly integrable.*

Proof. Let $K \subset L_1(\mu)$ be relatively weakly compact. Then K is bounded and if (f_n) is a sequence in K , then (f_n) has a weakly convergent subsequence.

Hence there is a subsequence (f_{n_j}) such that $\lim_j \int_E f_{n_j} d\mu$ exists for all $E \in \Sigma$.

It follows immediately that K is uniformly integrable.

Conversely, suppose K is bounded and uniformly integrable. Let (f_n) be a sequence in K . Then there is a countable field \mathcal{F} such that f_n is measurable relative to the σ -field Σ_1 , generated by \mathcal{F} .

By diagonal procedure, select a subsequence (f_{n_j}) such that $\lim_j \int_E f_{n_j} d\mu = F(E)$ exists for all $E \in \mathcal{F}$.

Since K is uniformly integrable, there exists $f \in L_1(\Sigma_1, \mu)$ such that

$$\lim_j \int_{\Omega} f_{n_j} g d\mu = \int_{\Omega} f g d\mu$$

for each $g \in L_{\infty}(\Sigma_1, \mu)$. Hence $f_{n_j} \rightarrow f$ is weakly in $L_1(\Sigma_1, \mu)$, But $f_{n_j} \rightarrow f$ is weakly $L_1(\mu)$, Hence K is relatively compact. \square

Theorem 2. *If $f : \Omega \rightarrow X^*$ is bounded weakly measure function, then $T_f : X^{**} \rightarrow L_1(\mu)$, is locally compact operator.*

Proof. Since $f : \Omega \rightarrow X^*$ is bounded there exists a number M such that $\sup\{\|f(x)\|; x \in \Omega\} \leq M$. If x^{**} belongs to B_X^{**} ,

$$\|T_f(x^{**})\| = \int_{\Omega} |x^{**} f| d\mu = \int_{\Omega} \|f(x)\| d\mu = M\mu(\Omega).$$

Hence $T_f(B_X^{**})$ is norm bounded. If $\varepsilon > 0$, $\mu(B) < \frac{\varepsilon}{M}$ then

$$\int_E \|f\| d\mu \leq M\mu(E) < \varepsilon \text{ and if } \mu(E) < \frac{\varepsilon}{M} \text{ and } x^{**} \in B_X^{**},$$

$$\int_E |T_f(x^{**})| d\mu = \int_E |x^{**} f| d\mu \leq \int_E \|f\| d\mu < \varepsilon$$

Hence $T_f(B_X^{**})$ is uniformly integrable. By Dunford theorem, $T_f(B_X^{**})$ is relatively weakly compact. Therefore $T_f : X^{**} \rightarrow L(\mu)$ is weakly compact operator. \square

Theorem 3. *Suppose that (Ω, Σ, μ) is a measure space, $f_n : \Omega \rightarrow X^*$ is bounded weak*-measurable for each $n \in N$, $\{f_n : n \in N\}$ is uniformly bounded and there is a real valued function gx on Ω such that $xf_n \rightarrow gx$ a.e. $[\mu]$. Then there is an $f : \Omega \rightarrow X^*$ such that $xf = gx$ a.e. $[\mu]$ for each $x \in X$.*

Proof. Suppose the hypothesis are satisfied. Let M_n be a sup $\{\|f_n(x^*)\| : x^* \in X^*\}$, since $\{f_n : n \in N\}$ is uniformly bounded, $M = \sup M_n < \infty$.

Let $K_M(0)$ denote the closed ball of radius M with center at the origin of X^* , the $K_M(0)$ is weak* compact and $(K_M(0), w^*)^\Omega$, there are a subset (f_{n_k}) of (f_n) and a function $f : \Omega \rightarrow K_M(0)$ such that (f_{n_k}) converges to f pointwise in the w^* -topology. But the $xf = gx$ a.e. $[\mu]$ for each $x \in X$. \square

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(BOK DONG YOO) NATIONAL TAX COLLEGE, SUWON, 440-747, KOREA