COHOMOLOGY GROUPS OF CIRCULAR UNITS

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Abstract. Let $k$ be a real abelian field of conductor $f$ and $k_{\infty} = \bigcup_{n \geq 0} k_n$ be its $\mathbb{Z}_p$-extension for an odd prime $p$ such that $p \nmid f\varphi(f)$. The aim of this paper is to compute the cohomology groups of circular units. For $m > n \geq 0$, let $G_{m,n}$ be the Galois group $\text{Gal}(k_m/k_n)$ and $C_m$ be the group of circular units of $k_m$. Let $l$ be the number of prime ideals of $k$ above $p$. Then, for $m > n \geq 0$, we have

1. Introduction

Let $k$ be a real abelian field of conductor $f$. For each prime $p$, let $k_{\infty} = \bigcup_{n \geq 0} k_n$ be the (basic) $\mathbb{Z}_p$-extension of $k$. For technical reasons, we will assume that $p$ is an odd prime such that $p \nmid f\varphi(f)$. Under this assumption, primes of $k$ above $p$ totally ramify in $k_n$ for all $n \geq 0$. Let $l$ be the number of prime ideals of $k$ above $p$.

For $m > n \geq 0$, we denote the Galois group $\text{Gal}(k_m/k_n)$ by $G_{m,n}$ and the norm map from $k_m$ to $k_n$ by $N_{m,n}$. Let $C_n$ be the group of circular units of $k_n$ as was defined by Sinnott([7]). The aim of this paper is to compute the following cohomology groups of $C_m$:

Theorem. For $m > n \geq 0$, we have

1. $C_{m,n}^{G_{m,n}} = C_n$,
2. $\hat{H}^i(G_{m,n}, C_m) \simeq (\mathbb{Z}/p^{m-n}\mathbb{Z})^{l-1}$ if $i$ is even,
3. $\hat{H}^i(G_{m,n}, C_m) \simeq (\mathbb{Z}/p^{m-n}\mathbb{Z})^l$ if $i$ is odd.


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Since $G_{m,n}$ is cyclic, $\hat{H}^i(G_{m,n}, C_m) \simeq \hat{H}^0(G_{m,n}, C_m) \simeq C_m^{G_{m,n}} / N_{m,n}C_m$ if $i$ is even, and $\hat{H}^i(G_{m,n}, C_m) \simeq \hat{H}^{-1}(G_{m,n}, C_m) \simeq N_{m,n}C_m / C_m^{G_{m,n}^{-1}}$ if $i$ is odd, where $\sigma_{m,n}$ is a generator of $G_{m,n}$. So it suffices to show (2) and (3) for $i = 0$ and $-1$, respectively.

Special cases such as when the conductor $f$ is a prime or divisible by two primes were studied in [5] and [1], respectively. The structure of circular units becomes much more complicated as $f$ has more distinct prime divisors. Thus the above theorem not only generalizes previous results but also shows that the cohomology groups of circular units are as simple as one can expect when compared to the cohomology groups of full units([3]).

The outline and the basic ideas of this paper are taken from [4], where cohomology groups of cyclotomic units are studied. And this paper generalizes the results of [4] about cyclotomic units in cyclotomic fields to circular units in abelian fields. The proof of (3) in the above theorem is especially similar to the corresponding one in [4], and so it could have been omitted. However we decided to include it for completeness.

2. Preliminaries and notations

In this section, we briefly review the definitions of cyclotomic units and circular units. Index theorems discovered by Sinnott are also introduced. Then we set up notations which will be used throughout this paper.

2.1. Cyclotomic units and circular units

Let $n \not\equiv 2 \pmod{4}$, $\zeta_n$ be an $n$th primitive root of 1, and $V$ be the multiplicative subgroup of $\mathbb{Q}(\zeta_n)^\times$ generated by

$$\{ \pm\zeta_n, 1 - \zeta_n^a \mid 1 \leq a \leq n - 1 \}.$$ 

Let $E$ be the group of units of $\mathbb{Q}(\zeta_n)$ and define $C = V \cap E$. $C$ is called the group of cyclotomic units of $\mathbb{Q}(\zeta_n)$. For $k = \mathbb{Q}(\zeta_n + \zeta_n^{-1})$, we define the group of cyclotomic units of $k$ by $C^+ = E^+ \cap C$, where $E^+$ is the group of units of $k$.

The group of cyclotomic units carries important information of $k$. To be precise, let $h^+$ be the class number of $k$. Then the index theorem of Sinnott([8]) says that $[E^+ : C^+] = 2^b h^+$ for some nonnegative integer $b$. This can be thought of as an algebraic version the analytic class number formula.
In general, for an abelian field $F$, Sinnott([8]) defines the group of circular units of $F$ as follows. For each $n > 2$, let

$$C_{F_n} = N_{Q(ζ_n)/F}N_{Q(ζ_n)}(C_{Q(ζ_n)})$$

where $C_{Q(ζ_n)}$ is the group of cyclotomic units of $Q(ζ_n)$. Then the group $C_F$ of circular units of $F$ is defined to be the multiplicative subgroup of $F^\times$ generated by $C_{F_n}$ for all $n > 2$ together with $-1$. Strictly speaking, the above definition is slightly different from the one given by Sinnott in [7]. But, since these two agree, we take the above one as the definition of circular units. Note that if $F \subset L$, then $N_{L/F}(C_L) \subset C_F \subset C_{Gal(L/F)}$, where $N_{L/F}$ is the norm map from $L$ to $F$. We will use this remark in the subsequent sections without comment. The index theorem of Sinnott([7]) says:

**Index Theorem.** Let $E_F$ be the unit group of $F$ and $h$ be the class number of the maximal real subfield of $F$. Then $[E_F : C_F] = c_F h$ for some integer $c_F$.

### 2.2. Notations

For each integer $s \geq 1$, we choose a primitive $s$th root $ζ_s$ of $1$ so that $ζ_s^s = ζ$. Let $k$ be a real abelian field of conductor $f$ and let $k_{∞} = \bigcup_{n \geq 0}k_{n}$ be its $Z_p$-extension. Note that $k$ admits a unique $Z_p$-extension for each prime $p$, namely the basic $Z_p$-extension. Throughout this paper, we assume that $p$ is an odd prime such that $p \nmid fϕ(f)$, where $ϕ$ is the Euler $ϕ$ function. For each $n \geq 0$, we denote the group of circular units of $k_{n}$ by $C_{n}$. Let $K = Q(ζ_f)$, $F = Q(ζ_p)$ and $K' = Q(ζ_{pf})$. We denote their basic $Z_p$-extensions by $K_{∞}$, $F_{∞}$, and $K'_{∞}$, respectively. The $n$th layers of these $Z_p$-extensions are denoted by $K_{n}$, $F_{n}$ and $K'_{n}$ as usual. Let $σ$ be the topological generator of the Galois group $Γ = Gal(K'_{∞}/K')$ which sends $ζ_{p^n}$ to $ζ_{p^{n+1}}$ for all $n \geq 1$. Restrictions of $σ$ to various subfields are also denoted by $σ$. We abbreviate $σ^{p^n}$ by $σ_n$. Thus $σ_n$ is a topological generator of $Gal(K'_{∞}/K'_n)$.

Let $k_{p}$ be the decomposition subfield of $k$ for $p$ and let $Δ = Gal(K/k)$, $Δ_{p} = Gal(K/k_{p})$, $Δ_k = Gal(k/Q)$ and $Δ_{k,p} = Gal(k_{p}/Q)$. Let $[k_{p} : Q] = l$, so there are $l$ prime ideals in $k$ above $p$. Elements of $Δ$, $Δ_{p}$ or $Δ_{k}$ will be denoted by $τ$’s and those of $Δ_k$ and $Δ_{k,p}$ by $ρ$’s. The Frobenius automorphism of $K$ for $p$ or its restriction to $k$ is denoted by $τ_p$. For each $d$ such that $d|f$, let $k_{(d)} = Q(ζ_d) \cap k$ and $k_{(p)}^{(d)}$ be the decomposition subfield of $k_{(d)}$ for $p$. Let $Δ_{(d)} = Gal(Q(ζ_d)/k_{(d)})$. 

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be the decomposition subfield of denoted by $ρ$'s. The Frobenius automorphism of $F$ as usual. Let $σ$ be the topological generator of the Galois group $Γ = Gal(K'_{∞}/K')$ which sends $ζ_{p^n}$ to $ζ_{p^{n+1}}$ for all $n \geq 1$. Restrictions of $σ$ to various subfields are also denoted by $σ$. We abbreviate $σ^{p^n}$ by $σ_n$. Thus $σ_n$ is a topological generator of $Gal(K'_{∞}/K'_n)$.

Let $k_{p}$ be the decomposition subfield of $k$ for $p$ and let $Δ = Gal(K/k)$, $Δ_{p} = Gal(K/k_{p})$, $Δ_k = Gal(k/Q)$ and $Δ_{k,p} = Gal(k_{p}/Q)$. Let $[k_{p} : Q] = l$, so there are $l$ prime ideals in $k$ above $p$. Elements of $Δ$, $Δ_{p}$ or $Δ_{k}$ will be denoted by $τ$’s and those of $Δ_k$ and $Δ_{k,p}$ by $ρ$’s. The Frobenius automorphism of $K$ for $p$ or its restriction to $k$ is denoted by $τ_p$. For each $d$ such that $d|f$, let $k_{(d)} = Q(ζ_d) \cap k$ and $k_{(p)}^{(d)}$ be the decomposition subfield of $k_{(d)}$ for $p$. Let $Δ_{(d)} = Gal(Q(ζ_d)/k_{(d)})$,
\[ \Delta^{(d)} = \text{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q}), \Delta^{(d)}_{(p)} = \text{Gal}(\mathbb{Q}(\zeta_d)/k^{(d)}_p), \Delta^{(d)}_k = \text{Gal}(k^{(d)}/\mathbb{Q}) \text{ and } \Delta^{(d)}_{k,p} = \text{Gal}(k^{(d)}_p/\mathbb{Q}). \]

Let \( R \) be the set of all roots of 1 in \( \mathbb{Z}_p \), i.e., \( R = \{ \omega \in \mathbb{Z}_p | \omega^{p-1} = 1 \} \). Then \( R \) can be regarded as the Galois group \( \text{Gal}(F/\mathbb{Q}) \) or as any Galois group isomorphic to it such as \( \text{Gal}(F_n/\mathbb{Q}_n) \), where \( \mathbb{Q}_n \) is the subfield of \( F_n \) of degree \( p^n \) over \( \mathbb{Q} \).

3. Computation of \( C_{m,n}^{G_{m,n}} \)

We keep all the previous notations. In this section, we will prove \( C_{m,n}^{G_{m,n}} = C_n \).

**Proof of Theorem (1).** Obviously, \( C_n \subset C_{m,n}^{G_{m,n}} \). To prove \( C_{m,n}^{G_{m,n}} \subset C_n \), take \( u \in C_{m,n}^{G_{m,n}} \). We will show that \( u^d \in C_n \) and \( u^{p^{m-n}} \in C_n \), where \( d = [\mathbb{Q}(\zeta_{pf}) : k] \). Then, since \( (d, p^{m-n}) = 1 \), we will have \( u \in C_n \). First, we view \( u \) as an element in \( \overline{C}_m^{G_{m,n}} \), where \( \overline{C}_m \) is the group of cyclotomic units of \( K'_s \) for each integer \( s \geq 0 \). Since \( \overline{C}_m^{G_{m,n}} = \overline{C}_n ([2]), \) \( u \) is a cyclotomic unit in \( K'_n \). Thus \( N_{K'_n/k_n}(u) \in C_n \). But since \( u \in K'_n \cap k_m \subset k_n \),

\[ N_{K'_n/k_n}(u) = u^d \in C_n. \]

Next, note that \( u^{p^{m-n}} = N_{m,n}(u) \) since \( u \) is fixed under \( G_{m,n} \). Thus

\[ u^{p^{m-n}} = N_{m,n}(u) \in C_n. \]

This finishes the proof. \( \square \)

4. Computation of \( \tilde{H}^0(G_{m,n}, C_m) \)

In this section we will prove

\[ \tilde{H}^0(G_{m,n}, C_m) \cong (\mathbb{Z}/p^{m-n}\mathbb{Z})^{1-1}. \]

First we need two lemmas.
Lemma 1. For \( m > n \geq 0 \), \( C_n = C_0 N_{m,n} C_m \).

Proof. Clearly, \( C_n \supset C_0 N_{m,n} C_m \). To prove the converse, note that an element \( u \) of \( C_n \) can be written as \( u = u_0 u_1 \cdots u_n \), where \( u_0 \in C_0 \), and for each \( s, 1 \leq s \leq n \), \( u_s \) is of the form

\[
\prod_{d|f} \prod_{1 \leq p^e \leq R} \prod_{\omega \in R} \sum_{\rho_j \in \Delta_k} u_s = \prod_{d|f} \prod_{1 \leq p^e \leq R} \prod_{\omega \in R} \sum_{\rho_j \in \Delta_k} u_s = N_{m,n} (\prod_{\omega \in R} (\zeta_{p^m+1} - \zeta_d^p)) = N_{m,n} (\prod_{\omega \in R} (\zeta_{p^m+1} - \zeta_d^p)),
\]

for some integers \( a_{i,j,d} \). Since \( N_{m,n} (\prod_{\omega, \tau} (\zeta_{p^m+1} - \zeta_d^p)) = \prod_{\omega, \tau} (\zeta_{p^m+1} - \zeta_d^p) \),

we have

\[
\prod_{\omega, \tau} (\zeta_{p^m+1} - \zeta_d^p) = \prod_{\omega, \tau} (\zeta_{p^{m-1}} - \zeta_d^p)^{\tau_{m-1}}
\]

\[
= N_{m,n} (\prod_{\omega, \tau} (\zeta_{p^m+1} - \zeta_d^p))^{\tau_{m-1}}
\]

\[
= N_{m,n} (\prod_{\omega, \tau} (\zeta_{p^m+1} - \zeta_d^p))^{\tau_{m-1}} \sum_{0 \leq \ell < p^{m-1}} \sigma(\ell).
\]

So for each \( s \geq 1 \), \( u_s \in N_{m,n} C_m \), and thus \( u \in C_0 N_{m,n} C_m \). This proves Lemma 1. \( \square \)

By rank \( A \) for a finitely generated abelian group \( A \), we mean the \( \mathbb{Z} \)-rank of the free part of \( A \).

Lemma 2. \( \text{rank } N_{k/k(p)} C_0 = l - 1 \).

Proof. Let \( C_{(p)} \) be the group of circular units of \( k_{(p)} \). Note that \( k^{(d)} \cap k_{(p)} = k_{(p)}^{(d)} \). Hence for \( \alpha \in k^{(d)} \),

\[
N_{k/k_{(p)}}(\alpha) = N_{k^{(d)}/k_{(p)}^{(d)}}(\alpha)[k^{(d)} : k_{(p)}].
\]

Therefore

\[
C_{(p)}^{\text{circ}}(k_{(p)}^{(f)}) \subset N_{k/k_{(p)}}(C_0) \subset C_{(p)} \subset E_{k_{(p)}},
\]

where \( E_{k_{(p)}} \) is the group of units of \( k_{(p)} \). Note that \( [E_{k_{(p)}} : C_{(p)}] \) is finite by the index theorem of Sinnott. Since \( C_{(p)} \) is a finitely generated
abelian group, \([C(p) : C_{p}^{(f)}]\) is also finite. Thus \(N_{k/k(p)}C_{0}\) is of finite index in the full unit group \(E_{k(p)}\). Therefore,

\[
\text{rank } N_{k/k(p)}C_{0} = \text{rank } E_{k(p)} = [k(p) : \mathbb{Q}] - 1 = l - 1.
\]

This finishes the proof of Lemma 2. 

Now we compute \(\hat{H}^{0}(G_{m,n}, C_{m})\).

**Proof of Theorem (2).** Since \(C_{n} = C_{0}N_{m,n}C_{m}\) by Lemma 1, the natural map

\[
C_{0} \rightarrow C_{n} \rightarrow C_{n}/N_{m,n}C_{m}
\]

is surjective. Thus

\[
\hat{H}^{0}(G_{m,n}, C_{m}) \simeq C_{m}^{G_{m,n}}/N_{m,n}C_{m} \simeq C_{n}/N_{m,n}C_{m} \simeq C_{0}/C_{0} \cap N_{m,n}C_{m}.
\]

Let \(C'_{m}\) be the subgroup of \(C_{m}\) generated by circular units of the form

\[
\prod_{\omega \in R, \tau \in \Delta(d)} (\zeta_{p^{m+1}}^{\omega} - \zeta_{d}^{\tau}),
\]

where \(p^{m+1} \nmid a\), and \(d \mid f\). Then clearly \(C_{m} = C_{0}C'_{m}\) and \(N_{m,n}C_{m}' = C_{n}'\). Hence \(N_{m,n}C_{m} = C_{0}^{p^{m-n}}C_{n}'\). Therefore

\[
\hat{H}^{0}(G_{m,n}, C_{m}) \simeq C_{0}/C_{0} \cap C_{0}^{p^{m-n}}C_{n}' = C_{0}/C_{0}^{p^{m-n}}(C_{0} \cap C_{n}').
\]

Next we claim that \(C_{0}^{p^{n-1}} \subset C_{0} \cap C_{0}' \subset N_{C_{0}}\), where \(N_{C_{0}} = \{ u \in C_{0} | N_{k/k(p)}u = 1 \}\). The first inclusion follows from the identity

\[
(1 - \zeta_{d})^{\tau - 1} = \prod_{\omega \in R} (\zeta_{p}^{\omega} - \zeta_{d}).
\]

To check the second one, take \(u \in C_{0} \cap C_{n}'\) and write \(u\) as

\[
u = \prod_{d | f, p^{m+1} \nmid a, b} \prod_{\omega \in R, \tau \in \Delta(d)} (\zeta_{p^{n+1}l_0}^{\omega} - \zeta_{d}^{\tau})^{g(a, b, d)}
\]

for some integers \(f(a, b, d)\). By taking \(N_{n,0}\), we have

\[
u^{p^{n}} = \prod_{e, d, \omega, \tau} (\zeta_{p}^{\omega} - \zeta_{d}^{\tau})^{g(e, d)} = \prod_{e, d, \tau} (1 - \zeta_{d}^{\tau})^{g(e, d)(\tau - 1)}
\]
for some integers $g(c, d)$. Therefore $N_{k/k_0} u^p^n = 1$ and the second inclusion follows. Since $N_{C_0}/C_0^{p^{n-1}}$ is annihilated by $[k : \mathbb{Q}]$, which is prime to $p$, we obtain

$$H^0(G_{m,n}, C_m) \simeq C_0/C_0^{p^{m-n}} (C_0 \cap C_n) = C_0/C_0^{p^{m-n}} N C_0.$$ 

For convenience, we denote $N_{k/k_0}$ simply by $N$. By Lemma 2, we know that $NC_0$ modulo $\{ \pm 1 \}$ is a free abelian group of rank $l - 1$. Let $\xi_1, \xi_2, \cdots, \xi_{l-1}$ be elements of $C_0$ such that $\{N(\xi_1), N(\xi_2), \cdots, N(\xi_{l-1})\}$ generates $NC_0$ modulo $\{ \pm 1 \}$. Let $C_0'$ be the subgroup of $C_0$ generated by $\{\xi_1, \xi_2, \cdots, \xi_{l-1}\}$. Then

$$[C_0 : C_0'' N C_0] = [NC_0 : NC_0'\cap NC_0 : NC_0] = 1 \text{ or } 2.$$ 

Therefore

$$H^0(G_{m,n}, C_m) \simeq \frac{\langle \xi_1, \cdots, \xi_{l-1}, \xi \rangle_{NC_0}}{\langle \xi_1, \cdots, \xi_{l-1}, \xi \rangle_{NC_0}^{p^{m-n}} N C_0} \simeq (\mathbb{Z}/p^{m-n}\mathbb{Z})^l$$

as desired. \hfill $\Box$

5. Computation of $\hat{H}^{-1}(G_{m,n}, C_m)$

In this section, we will prove

$$\hat{H}^{-1}(G_{m,n}, C_m) \simeq (\mathbb{Z}/p^{m-n}\mathbb{Z})^l.$$

**Lemma 3.** Let $n < m < s$. If $\delta \in C_m$ is such that $N_{m,n} \delta = 1$, then $\delta \in N_{s,m} C_s$.

**Proof.** Suppose $\delta \in C_m$ satisfies $N_{m,n} \delta = 1$. In Section 4, we proved that $C_0 = \langle \pm 1 \rangle (\xi_1, \xi_2, \cdots, \xi_{l-1})NC_0$. So $C_m = C_0 C_m' = \langle \pm 1 \rangle (\xi_1, \xi_2, \cdots, \xi_{l-1}) NC_0 C_m$. Thus we can write $\delta = \pm \xi uv$, for some $\xi \in \langle \xi_1, \xi_2, \cdots, \xi_{l-1} \rangle$, $u \in N C_0$ and $v \in C_m'$. Then $1 = N_{m,n} \delta = \pm \xi_{p^{m-n}} u v^{p^{m-n}} N_{m,n} v$. Since $v \in C_m'$, $N_{m,n} v \in C_m'$. Hence $N_{m,n} v = \pm \xi_{p^{m-n}} u v^{p^{m-n}} \in C_0 \cap C_m' \subset NC_0$. Therefore, we have $\xi_{p^{m-n}} w = \pm 1$, where $w = u v^{p^{m-n}} N_{m,n} v \in NC_0$. This implies that $\xi_{p^{m-n}} = \pm 1$, and thus we obtain $\xi = \pm 1$. Hence $\delta = \pm uv$.

Note that $u^t \in C_0 \cap C_m' \subset C_m'$, where $t = [N C_0 : C_0 \cap C_m']$, which is finite and prime to $p$. Therefore $\delta^{2t} = u^{2t} v^{2t} \in C_m' \subset N_{s,m} C_s$. Since $\delta^{p^{m-n}} = N_{s,m} (\delta) \in N_{s,m} C_s$, we have $\delta \in N_{s,m} C_s$, as desired. \hfill $\Box$
Proof of Theorem (3). We prove the theorem by induction on $m \geq n + 1$. Let $m = n + 1$. Since the Herbrand quotient for $C_m$ is $1/p(6)$ and since $p\tilde{H}^1(G_m, C_m) = 0$, $\tilde{H}^1(G_{m,n}, C_m)$ must be $(\mathbb{Z}/p\mathbb{Z})^l$.

Assuming the result for $m$, we will prove $\tilde{H}^1(G_{s,n}, C_s) = (\mathbb{Z}/p^{s-n}\mathbb{Z})^l$ when $s = m + 1$. Since the inflation map

$$\tilde{H}^1(G_{m,n}, C_m^{G_{s,m}}) \xrightarrow{\text{inflation}} \tilde{H}^1(G_{s,n}, C_s)$$

is injective, we may identify $\tilde{H}^1(G_{m,n}, C_m)$ with its image (we know that $C_{s}^{G_{s,m}} = C_m$). Let $[\delta]$ be an element in $\tilde{H}^{-1}(G_{m,n}, C_m)$. Since $\delta = N_{s,m} \xi$ for some $\xi \in C_s$ by Lemma 3, we have

$$[\xi^p] = [((N_{s,m} \xi)^p)^{N_{s,m} \xi}] = [N_{s,m} \xi] = [\delta].$$

Hence $\tilde{H}^1(G_{m,n}, C_m) \simeq \tilde{H}^{-1}(G_{m,n}, C_m) \simeq {N_{m,n} C_m}^1 / C_m^{n-1} = \mathbb{Z}/p^l$. Namely $\tilde{H}^{-1}(G_{m,n}, C_m) = p\tilde{H}^{-1}(G_{s,n}, C_s)$. Since $p^{s-n-1} \tilde{H}^{-1}(G_{s,n}, C_s) = 0$ and since the Herbrand quotient for $C_s$ is $p^{s-n}$, $\tilde{H}^1(G_{s,n}, C_s)$ must be $(\mathbb{Z}/p^{s-n}\mathbb{Z})^l$.

**Corollary.** Let $\Gamma = \text{Gal}(k_\infty/k)$ and $C_\infty = \bigcup_{n \geq 0} C_n$. Then

1. $H^1(\Gamma, C_\infty) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^l$
2. $H^2(\Gamma, C_\infty) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{l-1}$.

**Proof.** Since $C_n = C_{G_{m,n}}^m$ by Theorem (1), we have $C_n = C_{G_s}^s$, where $\Gamma_n = \text{Gal}(k_\infty/k_n)$. Hence, for $i = 1, 2$,

$$H^i(\Gamma, C_\infty) = \varinjlim H^i(G_n, C_n^{G_{m,n}})$$

$$= \varinjlim H^i(G_n, C_n)$$

$$\begin{cases} \varinjlim (\mathbb{Z}/p^\infty\mathbb{Z})^l & \text{if } i = 1 \\ \varinjlim (\mathbb{Z}/p^\infty\mathbb{Z})^{l-1} & \text{if } i = 2 \end{cases}$$

$$\begin{cases} (\mathbb{Q}_p/\mathbb{Z}_p)^l & \text{if } i = 1 \\ (\mathbb{Q}_p/\mathbb{Z}_p)^{l-1} & \text{if } i = 2. \end{cases}$$
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References


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