HYERS-ULAM-RASSIAS STABILITY OF A QUADRATIC TYPE FUNCTIONAL EQUATION

SANG HAN LEE AND KIL-WOUNG JUN

Abstract. In this paper, we prove the stability of a quadratic type functional equation

\[ a^2 f \frac{x+y+z}{a} + a^2 f \frac{x-y+z}{a} + a^2 f \frac{x+y-z}{a} + a^2 f \frac{-x+y+z}{a} = 4f(x) + 4f(y) + 4f(z). \]

1. Introduction

In 1940, S. M. Ulam ([9]) posed the following question on the stability of homomorphisms: Given a group \( G_1 \), a metric group \( G_2 \) with a metric \( d \) and a number \( \epsilon > 0 \), does there exist a \( \delta > 0 \) such that if a mapping \( f : G_1 \to G_2 \) satisfies the inequality

\[ d(f(xy), f(x)f(y)) < \delta \]

for all \( x, y \in G_1 \), then a homomorphism \( h : G_1 \to G_2 \) exists with

\[ d(f(x), h(x)) < \epsilon \]

for all \( x \in G_1 \)? This question became a source of the stability theory in the Hyers-Ulam sense. The case of approximately additive mappings was solved by D. H. Hyers ([1]) under the assumption that \( G_1 \) and \( G_2 \) are Banach spaces.

In 1978, Th. M. Rassias ([7]) generalized the result of Hyers as follows: Let \( f : X \to Y \) be a mapping between Banach spaces and let \( 0 \leq p < 1 \) be fixed. If \( f \) satisfies the inequality

\[ ||f(x+y) - f(x) - f(y)|| \leq \theta(||x||^p + ||y||^p) \]

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for some $\theta \geq 0$ and all $x, y \in X$, then there exists a unique additive mapping $A : X \rightarrow Y$ such that

$$||A(x) - f(x)|| \leq \frac{2\theta}{2 - 2p}||x||^p$$

for all $x \in X$. If, in addition, $f(tx)$ is continuous in $t$ for each fixed $x \in X$, then $A$ is linear.

The quadratic function $f(x) = x^2$ is a solution of the following functional equations

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

and

$$f(x + y + z) + f(x - y + z) + f(x + y - z) + f(-x + y + z) = 4f(x) + 4f(y) + 4f(z).$$

So, it is natural that each equation is called a quadratic functional equation. In particular, every solution of the quadratic functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is said to be a quadratic function.

In this paper we deal with a quadratic type functional equation

$$a^2 f \left( \frac{x + y + z}{a} \right) + a^2 f \left( \frac{x - y + z}{a} \right) + a^2 f \left( \frac{x + y - z}{a} \right)$$

$$+ a^2 f \left( \frac{-x + y + z}{a} \right) = 4f(x) + 4f(y) + 4f(z).$$

Throughout this paper $a$ is a nonzero real constant.

2. Solutions of a quadratic type functional equation

Throughout this section $X$ and $Y$ will be real linear spaces. Given a function $f : X \rightarrow Y$, consider the following equation

(2.1)

$$a^2 f \left( \frac{x + y + z}{a} \right) + a^2 f \left( \frac{x - y + z}{a} \right) + a^2 f \left( \frac{x + y - z}{a} \right)$$

$$+ a^2 f \left( \frac{-x + y + z}{a} \right) = 4f(x) + 4f(y) + 4f(z).$$
Lemma 1. If an even function $f : X \to Y$ satisfies (2.1) for all $x, y, z \in X$ and $f(0) = 0$, then $f$ is quadratic.

Proof. Note that $f(-x) = f(x)$ for all $x \in X$ since $f$ is an even function. Putting $y = z = 0$ in (2.1) we have

$$a^2 f \left( \frac{x}{a} \right) = f(x)$$

for all $x \in X$. Using (2.2) in (2.1) we have

$$f(x + y + z) + f(x - y + z) + f(x + y - z) + f(-x + y + z)$$

$$= 4f(x) + 4f(y) + 4f(z)$$

for all $x, y, z \in X$. Putting $z = 0$ in (2.3) we deduce $2f(x) + 2f(y) = f(x + y) + f(x - y)$ for all $x, y \in X$. This shows that $f$ is quadratic. \( \square \)

Lemma 2. If an odd function $f : X \to Y$ satisfies (2.1) for all $x, y, z \in X$, then $f$ is additive.

Proof. Note that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$ since $f$ is an odd function. Putting $y = z = 0$ in (2.1) we have

$$a^2 f \left( \frac{x}{a} \right) = 2f(x)$$

for all $x \in X$. Using (2.4) in (2.1) we have

$$f(x + y + z) + f(x - y + z) + f(x + y - z) + f(-x + y + z)$$

$$= 2f(x) + 2f(y) + 2f(z)$$

for all $x, y, z \in X$. Putting $z = 0$ in (2.5) we deduce $f(x + y) = f(x) + f(y)$ for all $x, y \in X$. This shows that $f$ is additive. \( \square \)

Remark. In Lemma 2, an additive mapping $f$ is nonzero in general. But if $a$ is a rational number and $a \neq 2$ in (2.1), then $f \equiv 0$.

Theorem 3. If a function $f : X \to Y$ satisfies (2.1) for all $x, y, z \in X$ and $f(0) = 0$, then there exist an additive mapping $A : X \to Y$ and a quadratic function $Q : X \to Y$ such that

$$f(x) = Q(x) + A(x)$$

for all $x \in X$. 
Proof. Let \( A(x) := \frac{1}{2} (f(x) - f(-x)) \) for all \( x \in X \). Then \( A(-x) = -A(x) \) and \( A \) satisfies (2.1) for all \( x, y, z \in X \). By Lemma 2, \( A \) is additive.

Let \( Q(x) := \frac{1}{2} (f(x) + f(-x)) \) for all \( x \in X \). Then \( Q(0) = 0 \), \( Q(-x) = Q(x) \) and \( Q \) satisfies (2.1) for all \( x, y, z \in X \). By Lemma 1, \( Q \) is quadratic. Clearly, we have \( f(x) = Q(x) + A(x) \) for all \( x \in X \).

3. Stability of a quadratic type functional equation

Let \( \mathbb{R}_+ \) denote the set of nonnegative real numbers. Recall that a function \( H : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) is homogeneous of degree \( p > 0 \) if it satisfies \( H(tu, tv, tw) = t^p H(u, v, w) \) for all nonnegative real numbers \( t, u, v \) and \( w \). Throughout this section \( X \) and \( Y \) will be a real normed linear space and a real Banach space, respectively. We may assume that \( H \) is homogeneous of degree \( p \). Given a function \( f : X \to Y \), we set

\[
Df(x, y, z) := a^2 f \left( \frac{x + y + z}{a} \right) + a^2 f \left( \frac{x - y + z}{a} \right) + a^2 f \left( \frac{x + y - z}{a} \right) + a^2 f \left( \frac{-x + y + z}{a} \right) - 4f(x) - 4f(y) - 4f(z)
\]

for all \( x, y, z \in X \).

**Theorem 4.** Assume that \( \delta \geq 0 \), \( p \in (0, \infty) \setminus \{1\} \) and \( \delta = 0 \) when \( p > 1 \). Let an odd function \( f : X \to Y \) satisfy

\[
||Df(x, y, z)|| \leq \delta + H(||x||, ||y||, ||z||)
\]

for all \( x, y, z \in X \). Then there exists a unique additive mapping \( A : X \to Y \) such that

\[
||f(x) - A(x)|| \leq \frac{1}{2} \delta + \frac{1}{2 - 2p} h(x)
\]

for all \( x \in X \), where \( h(x) = \frac{1}{4} (H(||x||, ||x||, 0) + H(||2x||, 0, 0)) \).

**Proof.** Note that \( f(0) = 0 \) and \( f(-x) = -f(x) \) for all \( x \in X \) since \( f \) is an odd function. Putting \( y = z = 0 \) in (3.1) and then replacing \( x \) by \( 2x \) we have

\[
\left| a^2 f \left( \frac{2x}{a} \right) - 2f(2x) \right| \leq \frac{1}{2} (\delta + H(||2x||, 0, 0))
\]
for all $x \in X$. Putting $y = x$ and $z = 0$ in (3.1) we have

\[ a^2 f \left( \frac{2x}{a} \right) - 4f(x) \leq \frac{1}{2}(\delta + H(||x||, ||x||, 0)) \]  

for all $x \in X$. By (3.3) and (3.4), we have

\[ ||f(2x) - 2f(x)|| \leq \frac{1}{2}\delta + h(x) \]  

for all $x \in X$, where $h(x) = \frac{1}{4}(H(||x||, ||x||, 0) + H(||2x||, 0, 0))$.

We divide the remaining proof by two cases.

(I) The case $0 < p < 1$. By (3.5), we have

\[ ||f(x) - f(2x)|| \leq \frac{1}{4}\delta + \frac{1}{2}h(x) \]  

for all $x \in X$. Using (3.6) we have

\[ ||f(2^n x) - f(2^{n+1} x)|| = \frac{1}{2^n} ||f(2^nx) - \frac{f(2 \cdot 2^n x)}{2}|| \leq \frac{1}{2^n+2}\delta + \frac{1}{2}2^{(p-1)n}h(x) \]  

for all $x \in X$ and all positive integers $n$. By (3.7), we have

\[ ||f(2^m x) - f(2^n x)|| \leq \sum_{k=m}^{n-1} \frac{1}{2^{k+2}}\delta + \sum_{k=m}^{n-1} \frac{1}{2}2^{(p-1)k}h(x) \]  

for all $x \in X$ and all positive integers $m$ and $n$ with $m < n$. This shows that $\left\{ \frac{f(2^n x)}{2^n} \right\}$ is a Cauchy sequence for all $x \in X$ since the right side of (3.8) converges to zero when $m \to \infty$. Consequently, we can define a mapping $A : X \to Y$ by

\[ A(x) := \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \]  

for all $x \in X$. Since $f(-x) = -f(x)$ for all $x \in X$, we have $A(-x) = -A(x)$ for all $x \in X$. Also, we get

\[ ||DA(x, y, z)|| = \lim_{n \to \infty} 2^{-n} ||Df(2^n x, 2^ny, 2^n z)|| \leq \lim_{n \to \infty} 2^{-n}\delta + 2^{(p-1)n}H(||x||, ||y||, ||z||) = 0 \]
for all $x, y, z \in X$. By Lemma 2, it follows that $A$ is additive. By (3.6) and (3.7), we have

$$
(3.9) \quad \left\| \frac{f(2^n x)}{2^n} - f(x) \right\| \leq \sum_{k=0}^{n-1} \frac{1}{2k+2} \delta + \sum_{k=0}^{n-1} \frac{1}{2} 2^{(p-1)k} h(x)
$$

for all $x \in X$ and all positive integers $n$. Taking the limit in (3.9) as $n \to \infty$, we get (3.2).

Now, let $A' : X \to Y$ be another additive mapping satisfying (3.2). Then we have

$$
||A(x) - A'(x)|| = 2^{-n}||A(2^n x) - A'(2^n x)||
\leq 2^{-n}(||A(2^n x) - f(2^n x)|| + ||A'(2^n x) - f(2^n x)||)
\leq 2^{-n} \delta + \frac{2}{|2 - 2^p|} 2^{(p-1)n} h(x)
$$

for all $x \in X$ and all positive integers $n$. Since

$$
\lim_{n \to \infty} \left( 2^{-n} \delta + \frac{2}{|2 - 2^p|} 2^{(p-1)n} h(x) \right) = 0,
$$

we can conclude that $A(x) = A'(x)$ for all $x \in X$. This proves the uniqueness of $A$.

(II) The case $p > 1$. Replacing $x$ by $\frac{x}{2}$ in (3.5) we have

$$
(3.10) \quad \left\| 2f(2^{-1} x) - f(x) \right\| \leq 2^{-p} h(x)
$$

for all $x \in X$. Using (3.10) we have

$$
(3.11) \quad \left\| 2^n f(2^{-n} x) - 2^{n+1} f(2^{-(n+1)} x) \right\| \leq 2^{-p} 2^{(1-p)n} h(x)
$$

for all $x \in X$ and all positive integers $n$. By (3.10) and (3.11), we have

$$
\left\| 2^n f(2^{-n} x) - f(x) \right\| \leq \sum_{k=0}^{n-1} 2^{(1-p)k} 2^{-p} h(x)
$$

for all $x \in X$ and all positive integers $n$. The rest of the proof is similar to the corresponding part of the case $0 < p < 1$. \qed
**Theorem 5.** Assume that $\delta \geq 0$, $p \in (0, \infty) \setminus \{2\}$ and $\delta = 0$ when $p > 2$. Let an even function $f : X \to Y$ satisfy (3.1) for all $x, y, z \in X$ and $f(0) = 0$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$
(3.12) \quad ||f(x) - Q(x)|| \leq \frac{1}{4} \delta + \frac{1}{|4 - 2p|} h(x)
$$

for all $x \in X$, where $h(x) = \frac{1}{2} H(||x||, ||x||, 0) + \frac{1}{4} H(||2x||, 0, 0)$.

**Proof.** Putting $y = x$ and $z = 0$ in (3.1) we have

$$
(3.13) \quad \left| a^2 f \left( \frac{2x}{a} \right) - 4f(x) \right| \leq \frac{1}{2} \left( \delta + H(||x||, ||x||, 0) \right)
$$

for all $x \in X$. Putting $y = z = 0$ in (3.1) and then replacing $x$ by $2x$ we have

$$
(3.14) \quad \left| a^2 f \left( \frac{2x}{a} \right) - f(2x) \right| \leq \frac{1}{4} \left( \delta + H(||2x||, 0, 0) \right)
$$

for all $x \in X$. By (3.13) and (3.14), we have

$$
(3.15) \quad ||f(2x) - 4f(x)|| \leq \frac{3}{4} \delta + h(x)
$$

for all $x \in X$, where $h(x) = \frac{1}{2} H(||x||, ||x||, 0) + \frac{1}{4} H(||2x||, 0, 0)$.

We divide the remaining proof by two cases.

(I) The case $0 < p < 2$. By (3.15), we have

$$
(3.16) \quad \left| f(x) - \frac{f(2x)}{4} \right| \leq \frac{3}{16} \delta + \frac{1}{4} h(x)
$$

for all $x \in X$. Using (3.16) we have

$$
(3.17) \quad \left| \frac{f(2^n x)}{4^n} - \frac{f(2^{n+1} x)}{4^{n+1}} \right| = \frac{1}{4^n} \left| f(2^n x) - \frac{f(2 \cdot 2^n x)}{4} \right| \\
\leq \frac{3}{16} 4^{-n} \delta + \frac{1}{4} 2^{(p-2)n} h(x)
$$

for all $x \in X$ and all positive integers $n$. By (3.16) and (3.17), we have

$$
(3.18) \quad \left| \frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n} \right| \leq \sum_{k=m}^{n-1} \frac{3}{16} 4^{-k} \delta + \sum_{k=m}^{n-1} \frac{1}{4} 2^{(p-2)k} h(x)
$$
for all \( x \in X \) and all nonnegative integers \( m \) and \( n \) with \( m < n \). This shows that \( \left\{ \frac{f(2^n x)}{4^n} \right\} \) is a Cauchy sequence for all \( x \in X \). Consequently, we can define a function \( Q : X \to Y \) by

\[
Q(x) := \lim_{n \to \infty} \frac{f(2^n x)}{4^n}
\]

for all \( x \in X \). We have \( Q(0) = 0 \), \( Q(-x) = Q(x) \) and

\[
||DQ(x, y, z)|| = \lim_{n \to \infty} 4^{-n} ||Df(2^n x, 2^n y, 2^n z)||
\]

\[
\leq \lim_{n \to \infty} (4^{-n} \delta + 2^{(p-2)n} H(||x||, ||y||, ||z||))
\]

\[
= 0
\]

for all \( x, y, z \in X \). By Lemma 1, it follows that \( Q \) is quadratic. Putting \( m = 0 \) in (3.18) and letting \( n \to \infty \) we have (3.12). The proof of the uniqueness of \( Q \) is similar to the proof of Theorem 4.

(II) The case \( p > 2 \). Replacing \( x \) by \( \frac{x}{2} \) in (3.15) we have

\[
(3.19) \quad ||4f(2^{-1} x) - f(x)|| \leq 2^{-p} h(x)
\]

for all \( x \in X \). Using (3.19) we have

\[
(3.20) \quad ||4^n f(2^{-n} x) - 4^{n+1} f(2^{-(n+1)} x)|| \leq 2^{-p} 2^{(2-p)n} h(x)
\]

for all \( x \in X \). By (3.19) and (3.20), we have

\[
||4^n f(2^{-n} x) - f(x)|| \leq \sum_{k=0}^{n-1} 2^{(2-p)k} 2^{-p} h(x)
\]

for all \( x \in X \) and all positive integers \( n \). The rest of the proof is similar to the corresponding part of the case \( p < 2 \). \( \square \)

**Theorem 6.** Let \( \delta \geq 0 \) and \( p \in (0, \infty) \setminus \{1, 2\} \). Assume that \( \delta = 0 \) if \( p > 1 \) and \( ||(a^2 - 3)f(0)|| = 0 \) if \( p > 2 \). If a function \( f : X \to Y \) satisfy (3.1) for all \( x, y, z \in X \), then there exist a unique quadratic function \( Q : X \to Y \) and a unique additive mapping \( A : X \to Y \) such that

\[
(3.21) \quad ||f(x) - f(0) - Q(x) - A(x)||
\]

\[
\leq \frac{3}{4} \delta + ||(a^2 - 3)f(0)|| + \frac{1}{|4 - 2p|} h_1(x) + \frac{1}{|2 - 2p|} h_2(x),
\]
Thus we have the following corollaries.

\[
\left| \frac{f(x) + f(-x)}{2} - f(0) - Q(x) \right| \leq \frac{1}{4} \delta + ||(a^2 - 3)f(0)|| + \frac{1}{|4 - 2p|} h_1(x),
\]

and

\[
\left| \frac{f(x) - f(-x)}{2} - A(x) \right| \leq \frac{1}{2} \delta + \frac{1}{|2 - 2p|} h_2(x)
\]

for all \(x \in X\), where \(h_1(x) = \frac{1}{2} H(||x||, ||x||, 0) + \frac{1}{4} H(||2x||, 0, 0)\) and \(h_2(x) = \frac{1}{4} (H(||x||, ||x||, 0) + H(||2x||, 0, 0))\).

**Proof.** Let \(q_1(x) := \frac{1}{2}(f(x) + f(-x))\) for all \(x \in X\). Then \(q_1(0) = f(0), q_1(-x) = q_1(x)\) and

\[
||Dq_1(x, y, z)|| \leq \delta + H(||x||, ||y||, ||z||)
\]

for all \(x, y, z \in X\). By Theorem 5, there exists a unique quadratic function \(Q : X \rightarrow Y\) satisfying (3.22).

Let \(g(x) := \frac{1}{2}(f(x) - f(-x))\) for all \(x \in X\). Then \(g(-x) = -g(x)\) and

\[
||Dg(x, y, z)|| \leq \delta + H(||x||, ||y||, ||z||)
\]

for all \(x, y, z \in X\). By Theorem 4, there exists a unique additive mapping \(A : X \rightarrow Y\) satisfying (3.23). Clearly, we have (3.21) for all \(x \in X\). □

Define a function \(H : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+\) by \(H(a, b, c) = (a^p + b^p + c^p)\theta\) where \(\theta \geq 0\) and \(p \in (0, \infty)\). Then \(H\) is homogeneous of degree \(p\). Thus we have the following corollaries.

**Corollary 7.** Assume that \(\delta \geq 0, p \in (0, \infty) \setminus \{1\}\) and \(\delta = 0\) when \(p > 1\). Let an odd function \(f : X \rightarrow Y\) satisfy

\[
||Df(x, y, z)|| \leq \delta + \theta(||x||^p + ||y||^p + ||z||^p)
\]

for all \(x, y, z \in X\). Then there exists a unique additive mapping \(A : X \rightarrow Y\) such that

\[
||f(x) - A(x)|| \leq \frac{1}{2} \delta + \frac{2 + 2p}{4|2 - 2p|} \theta||x||^p
\]

for all \(x \in X\).
Corollary 8. Assume that $\delta \geq 0$, $p \in (0, \infty) \setminus \{2\}$ and $\delta = 0$ when $p > 2$. Let an even function $f : X \to Y$ satisfy

$$||Df(x, y, z)|| \leq \delta + \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $x, y, z \in X$ and $f(0) = 0$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$||f(x) - Q(x)|| \leq \frac{1}{4}\delta + \frac{4 + 2^p}{4|4 - 2^p|}\theta||x||^p$$

for all $x \in X$.

Corollary 9. Let $\delta \geq 0$ and $p \in (0, \infty) \setminus \{1, 2\}$. Assume that $\delta = 0$ if $p > 1$ and $||(a^2 - 3)f(0)|| = 0$ if $p > 2$. If a function $f : X \to Y$ satisfy

$$||Df(x, y, z)|| \leq \delta + \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $x, y, z \in X$, then there exist a unique quadratic function $Q : X \to Y$ and a unique additive mapping $A : X \to Y$ such that

$$||f(x) - f(0) - Q(x) - A(x)|| \leq \frac{3}{4}\delta + ||(a^2 - 3)f(0)|| + \left(\frac{4 + 2^p}{4|4 - 2^p|} + \frac{2 + 2^p}{4|2 - 2^p|}\right)\theta||x||^p;$$

$$\left|\left|\frac{f(x) + f(-x)}{2} - f(0) - Q(x)\right|\right| \leq \frac{1}{4}\delta + ||(a^2 - 3)f(0)|| + \frac{4 + 2^p}{4|4 - 2^p|}\theta||x||^p;$$

and

$$\left|\left|\frac{f(x) - f(-x)}{2} - A(x)\right|\right| \leq \frac{1}{2}\delta + \frac{2 + 2^p}{4|2 - 2^p|}\theta||x||^p$$

for all $x \in X$.

References


**Sang Han Lee**, Department of Cultural Studies, Chungbuk Provincial University of Science & Technology, Okcheon 373-807, Korea

*E-mail*: shlee@ctech.ac.kr

**Kil-Woung Jun**, Department of Mathematics, Chungnam National University, Taejeon 305-764, Korea

*E-mail*: kwjun@math.chungnam.ac.kr