MULTIPLIER TRANSFORMATIONS AND STRONGLY CLOSE-TO-CONVEX FUNCTIONS

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Abstract. The purpose of the present paper is to introduce some new subclasses of strongly close-to-convex functions in the open unit disk defined by multiplier transformations and study their properties. Our results include several previous known results as special cases.

1. Introduction

Let $A$ denote the class of analytic functions defined in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$ with the normalization $f(0) = f'(z) - 1 = 0$. If $f$ and $g$ are analytic in $U$, we say that $f$ is subordinate to $g$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $w$ in $U$ such that $f(z) = g(w(z))$. We denote by $\mathcal{S}^*(\eta)$ and $\mathcal{C}(\eta)$ the subclasses of $A$ consisting of all analytic functions which are, respectively, starlike and convex of order $\eta (0 \leq \eta < 1)$ in $U$. (see, e.g., Srivastava and Owa [16]).

If $f \in A$ satisfies

$$\left| \arg \left( \frac{zf'(z)}{f(z)} - \eta \right) \right| < \frac{\pi}{2} \beta \quad (z \in U)$$

for some $\eta (0 \leq \eta < 1)$ and $\beta (0 < \beta \leq 1)$, then $f$ is said to be strongly starlike of order $\beta$ and type $\eta$ in $U$. If $f \in A$ satisfies

$$\left| \arg \left( 1 + \frac{zf''(z)}{f'(z)} - \eta \right) \right| < \frac{\pi}{2} \beta \quad (z \in U)$$

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for some $\eta (0 \leq \eta < 1)$ and $\beta (0 < \beta \leq 1)$, then $f$ is said to be strongly convex of order $\beta$ and type $\eta$ in $\mathcal{U}$. We denote by $\mathcal{S}^\ast (\beta, \eta)$ and $\mathcal{C}(\beta, \eta)$ [6], respectively, the subclasses of $\mathcal{A}$ consisting of all strongly starlike and strongly convex of order $\beta$ and type $\eta$ in $\mathcal{U}$. It is obvious that $f \in \mathcal{A}$ belongs to $\mathcal{S}^\ast (\beta, \eta)$ if and only if $zf' \in \mathcal{S}^\ast (\beta, \eta)$. We also note that $\mathcal{S}^\ast (1, \eta) = \mathcal{S}^\ast (\eta)$ and $\mathcal{C}(1, \eta) = \mathcal{C}(\eta)$. In particular, the classes $\mathcal{S}^\ast (\beta, 0)$ and $\mathcal{C}(\beta, 0)$ have been extensively studied by Mocanu [8] and Nunokawa [11].

For any integer $n$, we define the multiplier transformations $I_n^\lambda$ of functions $f \in \mathcal{A}$ by

\[
I_n^\lambda f(z) = z + \sum_{k=2}^{\infty} k \left( \frac{1 + \lambda}{k + \lambda} \right)^n a_k z^k \quad (\lambda \geq 0).
\]

Obviously, we have

\[
I_n^\lambda (I_m^\lambda f(z)) = I_{n+m}^\lambda f(z)
\]

for all integers $m$ and $n$. The operators $I_n^\lambda$ are closely related to the Komatu integral operators [5] and the differential and integral operators defined by Salagean [13]. We also note that $I_0^\lambda f(z) = zf'(z)$ and $I_1^\lambda f(z) = f(z)$. Now we define new classes of analytic functions by using the multiplier transformations $I_n^\lambda$ defined by (1.1) as follows:

For any integer $n$, let $\mathcal{K}_n^\lambda (\gamma, \delta, \eta, A, B)$ be the class of functions $f \in \mathcal{A}$ satisfying the condition

\[
\left| \arg \left( \frac{z(I_n^\lambda f(z))'}{I_n^\lambda g(z)} - \gamma \right) \right| < \frac{\pi}{2} \quad (0 \leq \gamma < 1; 0 < \delta \leq 1; z \in \mathcal{U})
\]

for some $g \in \mathcal{S}_n^\lambda (\eta, A, B)$, where

\[
\mathcal{S}_n^\lambda (\eta, A, B) = \left\{ g \in \mathcal{A} : \frac{1}{1 - \eta} \left( \frac{zI_n^\lambda g(z)'}{I_n^\lambda g(z)} - \eta \right) < \frac{1 + Az}{1 + Bz} \right\}
\]

\[
(0 \leq \eta < 1; -1 \leq B < A \leq 1; z \in \mathcal{U})
\]

We note that $\mathcal{K}_0^\lambda (\gamma, 1, \eta, 1, -1)$ and $\mathcal{K}_1^\lambda (\gamma, 1, \eta, 1, -1)$ are the classes of quasi-convex and close-to-convex functions of order $\gamma$ and type $\eta$, respectively, introduced and studied by Noor and Alkhorasani [10] and Silverman [14]. Further, $\mathcal{K}_0^\lambda (0, \delta, 0, 1, -1)$ is the class of strongly close-to-convex functions of order $\delta$ in the sense of Pommerenke [12].
In the present paper, we give some argument properties of analytic functions belonging to $A$ which contain the basic inclusion relationships among the classes $K_n^\lambda(\gamma, \delta, \eta, A, B)$. The integral preserving properties in connection with the operator $I_n^\lambda$ defined by (1.1) are also considered. Furthermore, we obtain the previous results by Bernardi [1], Libera [4], Noor [9] and Noor and Alkhorasani [10] as special cases.

2. Main results

In proving our main results, we need the following lemmas.

**Lemma 2.1 [2].** Let $h$ be convex univalent in $U$ with $h(0) = 1$ and $\text{Re} (\beta h(z) + \gamma) > 0(\beta, \gamma \in \mathbb{C})$. If $p$ is analytic in $U$ with $p(0) = 1$, then

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \quad (z \in U)$$

implies

$$p(z) \prec h(z) \quad (z \in U).$$

**Lemma 2.2 [7].** Let $h$ be convex univalent in $U$ and $\omega$ be analytic in $U$ with $\text{Re} \omega(z) \geq 0$. If $p$ is analytic in $U$ and $p(0) = h(0)$, then

$$p(z) + \omega(z)zp'(z) \prec h(z) \quad (z \in U)$$

implies

$$p(z) \prec h(z) \quad (z \in U).$$

**Lemma 2.3 [11].** Let $p$ be analytic in $U$ with $p(0) = 1$ and $p(z) \neq 0$ in $U$. Suppose that there exists a point $z_0 \in U$ such that

\begin{align}
(2.1) & \quad \left| \arg p(z) \right| < \frac{\pi}{2} \alpha \text{ for } |z| < |z_0| \\
\text{and} \quad (2.2) & \quad \left| \arg p(z_0) \right| = \frac{\pi}{2} \alpha \quad (0 < \alpha \leq 1).
\end{align}

Then we have

\begin{align}
(2.3) & \quad \frac{z_0p'(z_0)}{p(z_0)} = ik\alpha,
\end{align}
where

\[(2.4)\quad k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \text{ when } \arg p(z_0) = \frac{\pi}{2} \alpha \]

\[(2.5)\quad k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \text{ when } \arg p(z_0) = -\frac{\pi}{2} \alpha \]

and

\[(2.6)\quad p(z_0)^{\frac{1}{a}} = \pm ia \ (a > 0). \]

At first, with the help of Lemma 2.1, we obtain the following

**Proposition 2.1.** Let \( h \) be convex univalent in \( U \) with \( h(0) = 1 \) and \( \Re h(z) > 0 \). If a function \( f \in A \) satisfies the condition

\[
\frac{1}{1 - \eta} \left( \frac{z(I^\lambda_n f(z))'}{I^\lambda_n f(z)} - \eta \right) \prec h(z) \quad (0 \leq \eta < 1; \ z \in U),
\]

then

\[
\frac{1}{1 - \eta} \left( \frac{z(I^\lambda_{n+1} f(z))'}{I^\lambda_{n+1} f(z)} - \eta \right) \prec h(z) \quad (0 \leq \eta < 1; \ z \in U).
\]

**Proof.** Let

\[
p(z) = \frac{1}{1 - \eta} \left( \frac{z(I^\lambda_{n+1} f(z))'}{I^\lambda_{n+1} f(z)} - \eta \right),
\]

where \( p \) is analytic function with \( p(0) = 1 \). By using the equation

\[(2.7)\quad z(I^\lambda_{n+1} f(z))' = (\lambda + 1)I^\lambda_n f(z) - \lambda I^\lambda_{n+1} f(z), \]

we get

\[(2.8)\quad \lambda + \eta + (1 - \eta)p(z) = (\lambda + 1) \frac{I^\lambda_n f(z)}{I^\lambda_{n+1} f(z)}. \]
Taking logarithmic derivatives in both sides of (2.8) and multiplying by \( z \), we have

\[
p(z) + \frac{zp'(z)}{\lambda + \eta + (1 - \eta)p(z)} = \frac{1}{1 - \eta} \left( \frac{z(I_n^\lambda f(z))'}{I_n^\lambda f(z)} - \eta \right) \quad (z \in \mathcal{U}).
\]

Applying Lemma 2.1, it follows that \( p \prec h \), that is,

\[
\frac{1}{1 - \eta} \left( \frac{z(I_n^\lambda f(z))'}{I_n^\lambda f(z)} - \eta \right) \prec h(z) \quad (z \in \mathcal{U}).
\]

□

Taking \( h(z) = (1 + Az)/(1 + Bz)(-1 \leq B < A \leq 1) \) in Proposition 2.1, we have

**Corollary 2.1.** The inclusion relation, \( S_\lambda^n(\eta, A, B) \subset S_\lambda^{n+1}(\eta, A, B) \), holds for any integer \( n \).

Letting \( n = \lambda = 0 \) and \( h(z) = ((1 + z)/(1 - z))^\beta \) \((0 < \beta \leq 1)\) in Proposition 2.1, we have the following inclusion relation.

**Corollary 2.2.** \( C(\beta, \eta) \subset S^*(\beta, \eta) \).

**Proposition 2.2.** Let \( h \) be convex univalent in \( \mathcal{U} \) with \( h(0) = 1 \) and \( \text{Re} \ h(z) > 0 \). If a function \( f \in \mathcal{A} \) satisfies the condition

\[
\frac{1}{1 - \eta} \left( \frac{z(I_n^\lambda f(z))'}{I_n^\lambda f(z)} - \eta \right) \prec h(z) \quad (0 \leq \eta < 1; \ z \in \mathcal{U}),
\]

then

\[
\frac{1}{1 - \eta} \left( \frac{z(I_n^\lambda F_c(f)(z))'}{I_n^\lambda F_c(f)(z)} - \eta \right) \prec h(z) \quad (0 \leq \eta < 1; \ z \in \mathcal{U}),
\]

where \( F \) be the integral operator defined by

\[
F_c(f) := F_c(f)(z) = \frac{c + 1}{z^c} \int_0^z t^{c-1} f(t)\,dt \quad (c \geq 0).
\]

**Proof.** From (2.9), we have

\[
z(I_n^\lambda F_c(f)(z))' = (c + 1)I_n^\lambda f(z) - cI_n^\lambda F_c(f)(z).
\]
Let
\[ p(z) = \frac{1}{1 - \eta} \left( \frac{z(I_n^\lambda F_c(f)(z))'}{I_n^\lambda F_c(f)(z)} - \eta \right), \]
where \( p \) is analytic function with \( p(0) = 1 \). Then, by using (2.10), we get
\[ (2.11) \quad c + \eta + (1 - \eta)p(z) = (c + 1) \frac{I_n^\lambda f(z)}{I_n^\lambda F_c(f)(z)}. \]

Taking logarithmic derivatives in both sides of (2.11) and multiplying by \( z \), we have
\[ p(z) + \frac{zp'(z)}{c + \eta + (1 - \eta)p(z)} = \frac{1}{1 - \eta} \left( \frac{z(I_n^\lambda f(z))'}{I_n^\lambda F_c(f)(z)} - \eta \right) \quad (z \in \mathcal{U}). \]

Therefore, by Lemma 2.1, we have
\[ \frac{1}{1 - \eta} \left( \frac{z(I_n^\lambda f_c(f)(z))'}{I_n^\lambda F_c(f)(z)} - \eta \right) \prec h(z) \quad (z \in \mathcal{U}). \]

Letting \( h(z) = (1 + Az)/(1 + Bz)(-1 \leq B < A \leq 1) \) in Proposition 2.2, we have immediately

**COROLLARY 2.3.** If \( f \in S_n^\lambda(\eta, A, B) \), then \( F_c(f) \in S_n^\lambda(\eta, A, B) \), where \( F_c \) is the integral operator defined by (2.6).

**REMARK 2.1.** If we take \( h(z) = ((1 + z)/(1 - z))^\beta \) \((0 < \beta \leq 1) \) in Proposition 2.2, we see immediately that all functions belonging to the classes \( S^\star(\beta, \eta) \) and \( C^\star(\beta, \eta) \), respectively, preserve the angles under the integral operator defined by (2.9).

Now, we derive

**THEOREM 2.1.** Let \( f \in \mathcal{A} \) and \( 0 < \delta \leq 1, 0 \leq \gamma < 1 \). If
\[ \left| \arg \left( \frac{z(I_n^\lambda f(z))'}{I_n^\lambda g(z)} - \gamma \right) \right| < \frac{\pi}{2} \delta \]
for some \( g \in S_n^\lambda(\eta, A, B) \), then
\[ \left| \arg \left( \frac{z(I_{n+1}^\lambda f(z))'}{I_{n+1}^\lambda g(z)} - \gamma \right) \right| < \frac{\pi}{2} \alpha, \]
where \( \alpha (0 < \alpha \leq 1) \) is the solution of the equation:

\[
\delta = \begin{cases} 
\alpha + \frac{2}{\pi} \tan^{-1} \left( \frac{\alpha \cos \frac{\pi}{2} t_1}{(1-\eta)(1+A) + \eta + \lambda + \alpha \sin \frac{\pi}{2} t_1} \right) & \text{for } B \neq -1, \\
\alpha & \text{for } B = -1,
\end{cases}
\]

and

\[
t_1 = \frac{2}{\pi} \sin^{-1} \left( \frac{(1-\eta)(A-B)}{(1-\eta)(1-AB) + (\eta + \lambda)(1-B^2)} \right).
\]

**Proof.** Let

\[
p(z) = \frac{1}{1-\gamma} \left( \frac{z(I_{n+1}^\lambda f(z))'}{I_{n+1}^\lambda g(z)} - \gamma \right).
\]

Using (2.7) and simplifying, we have

\[
((1-\gamma)p(z) + \gamma)I_{n+1}^\lambda g(z) = (\lambda + 1)I_n^\lambda f(z) - \lambda I_{n+1}^\lambda f(z).
\]

Differentiating (2.14) and multiplying by \( z \), we obtain

\[
(1-\gamma)zp'(z)I_{n+1}^\lambda g(z) + ((1-\gamma)p(z) + \gamma)z(I_{n+1}^\lambda g(z))' \\
= (\lambda + 1)z(I_n^\lambda f(z))' - \lambda z(I_{n+1}^\lambda f(z))'.
\]

Since \( g \in S_\lambda^\lambda(\eta, A, B) \), by Corollary 2.1, we know that \( g \in S_{n+1}^\lambda(\eta, A, B) \). Let

\[
q(z) = \frac{1}{1-\eta} \left( \frac{z(I_{n+1}^\lambda g(z))'}{I_{n+1}^\lambda g(z)} - \eta \right).
\]

Then, using (2.7) once again, we have

\[
(1-\eta)q(z) + \eta + \lambda = (\lambda + 1)I_n^\lambda g(z) I_{n+1}^\lambda g(z).
\]

From (2.15) and (2.16), we obtain

\[
\frac{1}{1-\gamma} \left( z \frac{I_n^\lambda f(z)}{I_n^\lambda g(z)} - \gamma \right) = p(z) + \frac{zp'(z)}{(1-\eta)q(z) + \eta + \lambda}.
\]
While, by using the result of Silverman and Silvia [15], we have

\[(2.17) \quad \left| \frac{q(z) - 1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2} \quad (z \in \mathcal{U}; \ B \neq -1)\]

and

\[(2.18) \quad \text{Re} \{q(z)\} > \frac{1 - A}{2} \quad (z \in \mathcal{U}; \ B = -1).\]

Then, from (2.17) and (2.18), we obtain

\[(1 - \eta)q(z) + \eta + \lambda = \rho e^{i\frac{\pi \phi}{2}},\]

where

\[
\left\{ \begin{array}{l}
\frac{(1 - \eta)(1 - A)}{1 - B} + \eta + \lambda < \rho < \frac{(1 - \eta)(1 + A)}{1 + B} + \eta + \lambda \\
-t_1 < \phi < t_1 \text{ for } B \neq -1,
\end{array} \right.
\]

when \(t_1\) is given by (2.11), and

\[
\left\{ \begin{array}{l}
\frac{(1 - \eta)(1 - A)}{2} + \eta + \lambda < \rho < \infty \\
-1 < \phi < 1 \text{ for } B = -1.
\end{array} \right.
\]

We note that \(p\) is analytic in \(\mathcal{U}\) with \(p(0) = 1\) and \(\text{Re} \ p(z) > 0\) in \(\mathcal{U}\) by applying the assumption and Lemma 2.2 with \(\omega(z) = 1/((1 - \eta)q(z) + \eta + \lambda)\). Hence \(p(z) \neq 0\) in \(\mathcal{U}\).

If there exists a point \(z_0 \in \mathcal{U}\) such that the conditions (2.1) and (2.2) are satisfied, then (by Lemma 2.3) we obtain (2.3) under the restrictions (2.4), (2.5) and (2.6).

At first, suppose that \(p(z_0)^{\frac{1}{\alpha}} = ia \ (a > 0)\). Then we obtain

\[
\arg \left( p(z_0) + \frac{z_1 p'(z_0)}{(1 - \eta)q(z_0) + \eta + \lambda} \right) = \frac{\pi}{2} \alpha + \arg \left( 1 + ia k(p e^{i\frac{\pi \phi}{2}})^{-1} \right) \\
\geq \frac{\pi}{2} \alpha + \tan^{-1} \left( \frac{\alpha k \sin \frac{\pi}{2}(1 - \phi)}{\rho + \alpha k \cos \frac{\pi}{2}(1 - \phi)} \right) \\
\geq \frac{\pi}{2} \alpha + \tan^{-1} \left( \frac{\alpha \cos \frac{\pi}{2} t_1}{(1 - \eta)(1 + A) + \eta + \lambda + \alpha \sin \frac{\pi}{2} t_1} \right) \\
= \frac{\pi}{2} \delta,
\]
where $\delta$ and $t_1$ are given by (2.12) and (2.13), respectively. Similarly, for the case $B = -1$, we have

$$\arg \left( p(z_0) + \frac{z_0 p'(z_0)}{(1 - \eta)q(z_0) + \eta + \lambda} \right) \geq \frac{\pi}{2} \alpha.$$  

These evidently contradict the assumption of Theorem 2.1.

Next, suppose that $p(z_0) \frac{\pi}{2} = -ia \ (a > 0)$. Applying the same method as the above, we have

$$\arg \left( p(z_0) + \frac{z_0 p'(z_0)}{(1 - \eta)q(z_0) + \eta + \lambda} \right) \leq -\frac{\pi}{2} \delta,$$

where $\delta$ and $t_1$ are given by (2.12) and (2.13), respectively. Similarly, for the case $B = -1$, we have

$$\arg \left( p(z_0) + \frac{z_1 p'(z_0)}{(1 - \eta)q(z_0) + \eta + \lambda} \right) \leq -\frac{\pi}{2} \alpha.$$

These also are contradiction to the assumption of Theorem 2.1. Therefore we complete the proof of Theorem 2.1.

From Theorem 2.1, we see easily the following

**Corollary 2.4.** The inclusion relation, $K^{\lambda}_{n}(\gamma, \delta, \eta, A, B) \subset K^{\lambda}_{n+1}(\gamma, \delta, \eta, A, B)$, holds for any integer $n$.

Taking $n = \lambda = 0$ in Theorem 2.1, we have

**Corollary 2.5.** Let $f \in A$. If

$$\left| \arg \left( \frac{(zf'(z))'}{g'(z)} - \gamma \right) \right| < \frac{\pi}{2} \delta \ (0 \leq \gamma < 1; \ 0 < \delta \leq 1)$$

for some $g \in S^{\lambda}_{0}(\eta, A, B)$, then

$$\left| \arg \left( \frac{zf'(z)}{g(z)} - \gamma \right) \right| < \frac{\pi}{2} \alpha,$$
where $\alpha(0 < \alpha \leq 1)$ is the solution of the equation given by (2.12).

**Remark 2.2.** If we put $A = 1$, $B = -1$ and $\delta = 1$ in Corollary 2.5, then we see that every quasi-convex function of order $\gamma$ and type $\eta$ is close-to-convex function of order $\gamma$ and type $\eta$, which reduces the result obtained by Noor [9].

Letting $n = \lambda = \gamma = 0$, $B \to A(A < 1)$ and $g(z) = z$ in Theorem 2.1, we obtain

**Corollary 2.6.** Let $f \in A$ and $0 < \delta \leq 1$. If

$$| \arg (f'(z) + zf''(z)) | < \frac{\pi}{2}\delta,$$

then

$$| \arg f'(z) | < \frac{\pi}{2}\alpha,$$

where $\alpha (0 < \alpha \leq 1)$ is the solution of the equation:

$$\delta = \alpha + \frac{2}{\pi} \tan^{-1} \alpha.$$

Next, we prove

**Theorem 2.2.** Let $f \in A$ and $0 < \delta \leq 1$, $0 \leq \gamma < 1$. If

$$| \arg \left( \frac{z(I_n^\lambda f(z))'}{I_n^\lambda g(z)} - \gamma \right) | < \frac{\pi}{2}\delta$$

for some $g \in S_n^{\lambda}(\eta, A, B)$, then

$$| \arg \left( \frac{z(I_n^\lambda F_c(f))(z)'}{I_n^\lambda F_c(g)(z)} - \gamma \right) | < \frac{\pi}{2}\alpha,$$

where $F_c$ is defined by (2.9), and $\alpha (0 < \alpha \leq 1)$ is the solution of the equation given by (2.12).

**Proof.** Let

$$p(z) = \frac{1}{1 - \gamma} \left( \frac{z(I_n^\lambda F_c(f)(z))'}{I_n^\lambda F_c(g)(z)} - \gamma \right).$$
Since \( g \in S^{\lambda}_{n}(\eta, A, B) \), we have from Proposition 2.2 that \( F_{c}(g) \in S^{\lambda}_{n}(\eta, A, B) \). Using (2.7) we have

\[
(1 - \gamma)p(z) + \gamma I^{\lambda}_{n}F_{c}(g)(z) = (c + 1)I^{\lambda}_{n}f(z) - cI^{\lambda}_{n}F_{c}(f)(z).
\]

Then, by a simple calculation, we get

\[
(1 - \gamma)zp'(z) + ((1 - \gamma)p(z) + \gamma)((1 - \eta)q(z) + c + \eta) = (c + 1)\frac{z(I^{\lambda}_{n}F_{c}(g)(z))'}{I^{\lambda}_{n}F_{c}(g)(z)},
\]

where

\[
q(z) = \frac{1}{1 - \eta} \left( \frac{z(I^{\lambda}_{n}F_{c}(g)(z))'}{I^{\lambda}_{n}F_{c}(g)(z)} - \gamma \right).
\]

Hence we have

\[
\frac{1}{1 - \gamma} \left( \frac{z(I^{\lambda}_{n}f(z))'}{I^{\lambda}_{n}f(z)} - \gamma \right) = p(z) + \frac{zp'(z)}{(1 - \eta)q(z) + \eta + c}.
\]

The remaining part of the proof in Theorem 2.2 is similar to that of Theorem 2.1 and so we omit it. \(\square\)

From Theorem 2.2, we see easily the following

**Corollary 2.7.** If \( f \in K^{\lambda}_{n}(\gamma, \delta, \eta, A, B) \), then \( F_{c}(f) \in K^{\lambda}_{n}(\gamma, \delta, \eta, A, B) \), where \( F_{c} \) is the integral operator defined by (2.9).

**Remark 2.3.** If we take \( n = \lambda = 0 \) and \( n = 1, \lambda = 0 \) with \( \delta = 1, A = 1 \) and \( B = -1 \) in Corollary 2.7, respectively, then we have the corresponding results obtained by Noor and Alkhorasani [10]. Furthermore, taking \( n = 1, \gamma = \lambda = 0, A = 1, B = -1 \) and \( \delta = 1 \) in Corollary 2.7, we obtain the classical result by Bernardi [1], which implies the result studied by Libera [4].

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**References**


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