WEAK LAWS FOR WEIGHTED SUMS OF RANDOM VARIABLES

Soo Hak Sung

Abstract. Let \( \{a_{ni}, u_n \leq i \leq v_n, n \geq 1\} \) be an array of constants. Let \( \{X_{ni}, u_n \leq i \leq v_n, n \geq 1\} \) be an array of random variables. \( \{a_{ni}\} \)-uniformly integrable random variables. Weak laws for the weighted sums \( \sum_{i=u_n}^{v_n} a_{ni} X_{ni} \) are obtained.

1. Introduction

Let \( \{u_n, n \geq 1\} \) and \( \{v_n, n \geq 1\} \) be two sequences of integers (not necessarily positive or finite), and let \( \{k_n, n \geq 1\} \) be a sequence of positive integers such that \( k_n \to \infty \) as \( n \to \infty \). Consider an array of constants \( \{a_{ni}, u_n \leq i \leq v_n, n \geq 1\} \) and an array of random variables \( \{X_{ni}, u_n \leq i \leq v_n, n \geq 1\} \) defined on a probability space \((\Omega, \mathcal{F}, P)\).

When \( u_n = 1, v_n = k_n, n \geq 1 \), weak laws of large numbers for the array \( \{X_{ni}\} \) have been established by several authors (see, Gut [2], Hong and Lee [3], Hong and Oh [4], and Sung [8]). An array \( \{X_{ni}, 1 \leq i \leq k_n, n \geq 1\} \) is said to be Cesàro uniformly integrable if

\[
\lim_{a \to \infty} \sup_{n \geq 1} \frac{1}{k_n} \sum_{i=1}^{k_n} E|X_{ni}| I(|X_{ni}| > a) = 0.
\]

This condition was introduced by Chandra [1].

Ordonez Cabrera [5] extended the notion of Cesàro uniform integrability to \( \{a_{ni}\} \)-uniform integrability as follows.
An array of random variables \( \{X_{ni}, u_n \leq i \leq v_n, n \geq 1\} \) is said to be \( \{a_{ni}\} \)-uniformly integrable if

\[
\lim_{a \to \infty} \sup_{n \geq 1} \sum_{i=u_n}^{v_n} |a_{ni}| E|X_{ni}| I(|X_{ni}| > a) = 0.
\]

Note that \( \{a_{ni}\} \)-uniform integrability reduces to Cesàro uniform integrability when \( a_{ni} = k_n^{-1} \) for \( 1 \leq i \leq k_n \) and 0 elsewhere. Ordonez Cabrera [5] obtained weak laws for weighted sums of \( \{a_{ni}\} \)-uniformly integrable random variables, where the weights satisfy

\[
\lim_{n \to \infty} \sup_{u_n \leq i \leq v_n} |a_{ni}| = 0 \quad \text{and} \quad \sup_{n \geq 1} \sum_{i=u_n}^{v_n} |a_{ni}|^r < \infty \quad \text{for some} \quad 0 < r \leq 1.
\]

In this paper, we prove the results of Ordonez Cabrera [5] under a weaker condition on the weights.

2. Main result

Throughout this section, let \( \{a_{ni}, u_n \leq i \leq v_n, n \geq 1\} \) be an array of constants, and let \( \{X_{ni}, u_n \leq i \leq v_n, n \geq 1\} \) be an array of random variables.

To prove the main result, we will need the following lemmas.

**Lemma 1.** (Sung [9]). Suppose that \( \{X_{ni}, u_n \leq i \leq v_n, n \geq 1\} \) is an array of random variables satisfying the following conditions.

\[
\sup_{n \geq 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} E|X_{ni}|^r < \infty
\]

and

\[
\lim_{a \to \infty} \sup_{n \geq 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} E|X_{ni}|^r I(|X_{ni}|^r > a) = 0,
\]

where \( r > 0 \) and \( \{k_n, n \geq 1\} \) is a sequence of positive numbers such that \( k_n \to \infty \) as \( n \to \infty \). If \( \beta > r \), then

\[
\sum_{i=u_n}^{v_n} E|X_{ni}|^\beta I(|X_{ni}|^r \leq k_n) = o(k_n^{\beta/r}).
\]
Lemma 2. (Sung [9]). Suppose that \( \{X_{ni}, u_n \leq i \leq v_n, n \geq 1\} \) is an array of random variables satisfying (3) and (4) for some \( 0 < r < 1 \) and \( \{k_n, n \geq 1\} \). Then
\[
\frac{\sum_{i=u_n}^{v_n} X_{ni}}{k_n^{1/r}} \rightarrow 0
\]
in \( L^r \) and, hence, in probability as \( n \rightarrow \infty \).

Lemma 3. Let \( r > 0 \). Let \( \{|X_{ni}|^r, u_n \leq i \leq v_n, n \geq 1\} \) be \( \{|a_{ni}|^r\} \)-uniformly integrable random variables, where \( \{a_{ni}, u_n \leq i \leq v_n, n \geq 1\} \) is an array of constants satisfying \( \lim_{n \rightarrow \infty} \sup_{u_n \leq i \leq v_n} |a_{ni}| = 0 \). Assume that
\[
(5) \quad \sup_{n \geq 1} \sum_{i=u_n}^{v_n} |a_{ni}|^r E|X_{ni}|^r < \infty.
\]
If \( \beta > r \), then
\[
\sum_{i=u_n}^{v_n} |a_{ni}|^r E|X_{ni}|^\beta I(|X_{ni}|^r \leq k_n) = o(1),
\]
where \( k_n = 1/\sup_{u_n \leq i \leq v_n} |a_{ni}|^r \).

Proof. Take \( k_n^{1/r} a_{ni} X_{ni} \) instead of \( X_{ni} \) in Lemma 1. Then we have by (5) that
\[
\sup_{n \geq 1} \frac{1}{k_n^r} \sum_{i=u_n}^{v_n} E|k_n^{1/r} a_{ni} X_{ni}|^r < \infty.
\]
Since \( k_n |a_{ni}|^r \leq 1 \) for \( u_n \leq i \leq v_n \), it follows that
\[
\lim_{a \rightarrow \infty} \sup_{n \geq 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} E|k_n^{1/r} a_{ni} X_{ni}|^r I(|k_n^{1/r} a_{ni} X_{ni}|^r > a)
\]
\[
\leq \lim_{a \rightarrow \infty} \sup_{n \geq 1} \sum_{i=u_n}^{v_n} |a_{ni}|^r E|X_{ni}|^r I(|X_{ni}|^r > a) = 0.
\]
Thus we obtain by Lemma 1 that
\[
\frac{1}{k_n^{\beta/r}} \sum_{i=u_n}^{v_n} E|k_n^{1/r} a_{ni} X_{ni}|^\beta I(|k_n^{1/r} a_{ni} X_{ni}|^r \leq k_n) = o(1).
\]
So the result follows since \( k_n \leq 1/|a_{ni}|^r \) for \( u_n \leq i \leq v_n \). \( \square \)

Now, we state and prove our main result which generalizes some results in the literature in this area. See the corollaries and example following Theorem 1.
Theorem 1. Let $0 < r \leq 1$. Let \{a_{ni}, u_n \leq i \leq v_n, n \geq 1\} and \{X_{ni}, u_n \leq i \leq v_n, n \geq 1\} be as in Lemma 3. When $r = 1$, we assume further that \{X_{ni}\} is an array of (rowwise) pairwise independent random variables with $EX_{ni} = 0$, i.e., for each fixed $n$, $X_{n,u_n}, \ldots, X_{n,v_n}$ are pairwise independent. Then

$$\sum_{i=u_n}^{v_n} a_{ni}X_{ni} \to 0$$

in $L^r$ and, hence, in probability as $n \to \infty$.

Proof. Take $k_n = 1/\sup_{u_n \leq i \leq v_n} |a_{ni}|^r$ and $k_n^{1/r} a_{ni}X_{ni}$ instead of $X_{ni}$ in Lemma 2. When $0 < r < 1$, the result follows from Lemma 2.

We now prove the result when $r = 1$. Define $X'_{ni} = X_{ni}I(|X_{ni}| \leq k_n)$ and $X''_{ni} = X_{ni} - X'_{ni}$ for $u_n \leq i \leq v_n$ and $n \geq 1$. Since $X'_{n,u_n}, \ldots, X'_{n,v_n}$ are pairwise independent random variables, we have by Lemma 3 with $r = 1$ and $\beta = 2$ that

$$E\left|\sum_{i=u_n}^{v_n} a_{ni}(X'_{ni} - EX'_{ni})\right|^2 \leq \sum_{i=u_n}^{v_n} a_{ni}^2 E\left|X'_{ni}\right|^2 \to 0$$

as $n \to \infty$. Also we obtain by the definition of \{a_{ni}\}-uniform integrability that

$$E\left|\sum_{i=u_n}^{v_n} a_{ni}(X''_{ni} - EX''_{ni})\right| \leq 2 \sum_{i=u_n}^{v_n} |a_{ni}| E\left|X''_{ni}\right| \to 0$$

as $n \to \infty$, since $k_n \to \infty$ as $n \to \infty$. Thus we have

$$E\left|\sum_{i=u_n}^{v_n} a_{ni}X_{ni}\right|$$

$$\leq E\left|\sum_{i=u_n}^{v_n} a_{ni}(X'_{ni} - EX'_{ni})\right| + E\left|\sum_{i=u_n}^{v_n} a_{ni}(X''_{ni} - EX''_{ni})\right|$$

$$\leq (E\left|\sum_{i=u_n}^{v_n} a_{ni}(X'_{ni} - EX'_{ni})\right|^2)^{1/2} + E\left|\sum_{i=u_n}^{v_n} a_{ni}(X''_{ni} - EX''_{ni})\right| \to 0$$

as $n \to \infty$. □
Corollary 1. Let $0 < r < 1$. Let $\{ |X_{ni}|^r, u_n \leq i \leq v_n, n \geq 1 \}$ be $\{|a_{ni}|^r\}$-uniformly integrable random variables, where $\{a_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ is an array of constants satisfying \(\lim_{n \to \infty} \sup_{u_n \leq i \leq v_n} |a_{ni}| = 0\) and for some constant $C > 0$

\[
\sum_{i=1}^{v_n} |a_{ni}|^r < C \quad \text{for all } n. \tag{6}
\]

Then

\[
\sum_{i=u_n}^{v_n} a_{ni} X_{ni} \to 0
\]

in $L^r$ and, hence, in probability as $n \to \infty$.

Proof. From Theorem 1, it is enough to show that (5) holds. Since $\{ |X_{ni}|^r \}$ is $\{|a_{ni}|^r\}$-uniformly integrable, there exists $a > 0$ such that

\[
\sup_{n \geq 1} \sum_{i=1}^{v_n} |a_{ni}|^r E|X_{ni}|^r I(|X_{ni}|^r > a) \leq 1.
\]

Then

\[
E|X_{ni}|^r = E|X_{ni}|^r I(|X_{ni}|^r \leq a) + E|X_{ni}|^r I(|X_{ni}|^r > a) \\
\leq a + E|X_{ni}|^r I(|X_{ni}|^r > a),
\]

which implies by (6) that

\[
\sup_{n \geq 1} \sum_{i=1}^{v_n} |a_{ni}|^r E|X_{ni}|^r \\
\leq a \cdot \sup_{n \geq 1} \sum_{i=1}^{v_n} |a_{ni}|^r + \sup_{n \geq 1} \sum_{i=1}^{v_n} |a_{ni}|^r E|X_{ni}|^r I(|X_{ni}|^r > a) \\
\leq a \cdot C + 1.
\]

Hence (5) is satisfied. \qed

The above corollary has been proved by Ordonez Cabrera [5]. Rohatgi [7] established a weaker result (convergence in probability) under the stronger condition that $\{X_n, n \geq 1\}$ is a sequence of independent random variables which is uniformly bounded by a random variable $X$ with $E|X|^r < \infty$. Wang and Rao [10] extended Rohatgi’s result to $L^r$-convergence under the uniform integrability (without independent condition).
Corollary 2. Let \( \{X_{ni}, u_n \leq i \leq v_n, n \geq 1\} \) be \( \{a_{ni}\} \)-uniformly integrable (rowwise) pairwise independent random variables with \( EX_{ni} = 0 \) for \( u_n \leq i \leq v_n \) and \( n \geq 1 \), where \( \{a_{ni}, u_n \leq i \leq v_n, n \geq 1\} \) is an array of constants satisfying \( \lim_{n \to \infty} \sup_{u_n \leq i \leq v_n} |a_{ni}| = 0 \) and for some constant \( C > 0 \)

\[
\sum_{i=u_n}^{v_n} |a_{ni}| < C \text{ for all } n.
\]

Then

\[
\sum_{i=u_n}^{v_n} a_{ni}X_{ni} \to 0
\]

in \( L^1 \) and, hence, in probability as \( n \to \infty \).

Proof. By Theorem 1, it is enough to show that (5) holds when \( r = 1 \). The proof of the rest is similar to that of Corollary 1 and is omitted. \( \square \)

The above corollary has been proved by Ordonez Cabrera [5]. Pruitt [6] established a weaker result (convergence in probability) under the stronger condition that \( \{X_n, n \geq 1\} \) is a sequence of independent and identically distributed random variables with \( EX_n = 0 \) for \( n \geq 1 \). Rohatgi [7] extended Pruitt’s result to a sequence of independent random variables which is uniformly bounded by a random variable \( X \) with \( E|X| < \infty \). Wang and Rao [10] extended Rohatgi’s result to \( L^1 \)-convergence for uniformly integrable pairwise independent random variables.

The following example shows that the conditions of Theorem 1 are weaker than the conditions of Corollary 2.

Example 1. Let \( \{X_n, n \geq 1\} \) be a sequence of pairwise independent random variables such that \( X_n = \pm 1/\log n \) with probability \( 1/2 \) if \( n \) is not a perfect cube, and \( X_n = \pm n^{1/3} \) with probability \( 1/2 \) if \( n \) is a perfect cube (i.e., \( n = j^3 \) for some positive integer \( j \)). Define an array of constants \( \{a_{ni}, i \geq 1, n \geq 1\} \) as follows.

\[
a_{ni} = \begin{cases} 
\log n/n & \text{if } 1 \leq i \leq n, \\
0 & \text{if } i > n.
\end{cases}
\]

Since \( \sum_{i=1}^{\infty} |a_{ni}| = \log n \), we can not apply this example to Corollary 2. Observe that

\[
\sum_{i \neq j^3, i \leq n} E|X_i| = \sum_{i \neq j^3, i \leq n} 1/\log i = O(n/\log n)
\]
and
\[ \sum_{i=j^3, i \leq n} E|X_i| = \sum_{i=j^3, i \leq n} i^{1/3} = \sum_{j^3 \leq n} j = \frac{j_0(j_0 + 1)}{2} \leq \frac{n^{1/3}(n^{1/3} + 1)}{2}, \]

where \( j_0 = \max\{j : j^3 \leq n\} \). Thus we have
\[
\sum_{i=1}^{\infty} |a_{ni}|E|X_i| = \frac{\log n}{n} \sum_{i=1}^{n} E|X_i| \\
\leq \frac{\log n}{n} \left( O\left( \frac{n}{\log n} \right) + \frac{n^{1/3}(n^{1/3} + 1)}{2} \right) = O(1).
\]

If \( a > 1 \), then
\[
\sum_{i=1}^{\infty} |a_{ni}|E|X_i|I(|X_i| > a) = \frac{\log n}{n} \sum_{i=j^3, a^3 < i \leq n} E|X_i|.
\]

Hence, for \( a > 1 \), we have
\[
\sup_{n \geq 1} \sum_{i=1}^{\infty} |a_{ni}|E|X_i|I(|X_i| > a) = \sup_{n > a^3} \frac{\log n}{n} \sum_{i=j^3, a^3 < i \leq n} E|X_i| \\
\leq \sup_{n > a^3} \frac{n^{1/3}(n^{1/3} + 1) \log n}{2n} \to 0
\]
as \( a \to \infty \). Therefore the conditions of Theorem 1 with \( r = 1 \) are satisfied. By Theorem 1, we obtain
\[
\sum_{i=1}^{\infty} a_{ni}X_i \to 0
\]
in \( L^1 \) and, hence, in probability as \( n \to \infty \).

References


Department of Applied Mathematics, Pai Chai University, Taejon 302-735, Korea

E-mail: sungsh@mail.pcu.ac.kr