ON INTUITIONISTIC FUZZY R-SUBGROUPS OF NEAR-RINGS

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ABSTRACT. The notion of normal intuitionistic fuzzy R-subgroups in near-rings is introduced, and related properties are investigated. Characterization of an intuitionistic fuzzy R-subgroup is given. Using a collection of right R-subgroups, an intuitionistic fuzzy right R-subgroup is established. Using a chain of right R-subgroups, an intuitionistic fuzzy right R-subgroup is also established.

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1. Introduction

After the introduction of fuzzy sets by Zadeh [7], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [1] is one among them. For the details on intuitionistic fuzzy sets, we refer the reader to [2, 3]. Jun et al. [5] introduced the notion of an intuitionistic fuzzy ideal of a near-ring, and investigated some properties. Yon et al. [6] considered the intuitionistic fuzzification of a right (resp. left) R-subgroup of a near-ring. In this paper we investigate more properties of an intuitionistic fuzzy right R-subgroup, and introduce the notion of a normal intuitionistic fuzzy right R-subgroup. We give a characterization of an intuitionistic fuzzy R-subgroup. Using a collection of right R-subgroups [4], we establish an intuitionistic fuzzy right R-subgroup. Using a chain of right R-subgroups, we also construct an intuitionistic fuzzy right R-subgroup.
2. Preliminaries

By a near-ring we mean a non-empty set $R$ with two binary operations \(\cdot\) and \(\cdot\) satisfying the following axioms:

(i) \((R, +)\) is a group,
(ii) \((R, \cdot)\) is a semigroup,
(iii) \(x \cdot (y + z) = x \cdot y + x \cdot z\) for all \(x, y, z \in R\).

Precisely speaking, it is a left near-ring because it satisfies the left distributive law. We will use the word “near-ring” instead of “left near-ring”. We denote \(xy\) instead of \(x \cdot y\). Note that \(x0 = 0\) and \(x(-y) = -xy\) but in general \(0x \neq 0\) for some \(x \in R\). A two-sided \(R\)-subgroup of a near-ring \(R\) is a subset \(H\) of \(R\) such that

(i) \((H, +)\) is a subgroup of \((R, +)\),
(ii) \(RH \subseteq H\),
(iii) \(HR \subseteq H\).

If \(H\) satisfies (i) and (ii) then it is called a left \(R\)-subgroup of \(R\). If \(H\) satisfies (i) and (iii) then it is called a right \(R\)-subgroup of \(R\). Let \(S\) be a nonempty set and let \(\mu_A\) and \(\gamma_A\) be two functions from \(S\) to \([0, 1]\) such that

\[
(\forall x \in S) \left( 0 \leq \mu_A(x) + \gamma_A(x) \leq 1 \right).
\]

By the original definition of Atanassov in [1], an intuitionistic fuzzy set (IFS for short) is an object of the form: \(A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in S\}\). We consider it in a form of an ordered triple: \(A = \langle R; \mu_A, \gamma_A \rangle\) where \(\mu_A\) and \(\gamma_A\) are as above. An IFS \(A = \langle R; \mu_A, \gamma_A \rangle\) in a near-ring \(R\) is called an intuitionistic fuzzy subnear-ring of \(R\) (see [6]) if it satisfies

\[
\begin{align*}
\forall x, y \in R &\left( \mu_A(x - y) \geq \min\{\mu_A(x), \mu_A(y)\} \right), \\
\forall x, y \in R &\left( \gamma_A(x - y) \leq \max\{\gamma_A(x), \gamma_A(y)\} \right), \\
\forall x, y \in R &\left( \mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\} \right), \\
\forall x, y \in R &\left( \gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\} \right).
\end{align*}
\]

3. (Normal) intuitionistic fuzzy \(R\)-subgroups

**Definition 3.1.** [6] An IFS \(A = \langle R; \mu_A, \gamma_A \rangle\) in a near-ring \(R\) is called an intuitionistic fuzzy right \(R\)-subgroup of \(R\) if it satisfies

\[
\begin{align*}
\forall x, y \in R &\left( \mu_A(x - y) \geq \min\{\mu_A(x), \mu_A(y)\} \right), \\
\forall x, y \in R &\left( \gamma_A(x - y) \leq \max\{\gamma_A(x), \gamma_A(y)\} \right).
\end{align*}
\]
Let \( A \) be an intuitionistic fuzzy right \( R \)-subgroup of a near-ring \( R \). Then the set
\[
R_A := \{ x \in R \mid \mu_A(x) = \mu_A(0), \gamma_A(x) = \gamma_A(0) \}
\]
is a right \( R \)-subgroup of \( R \).

Proof. Let \( x, y \in R \). Then \( \mu_A(x) = \mu_A(y) = \mu_A(0) \) and \( \gamma_A(x) = \gamma_A(y) = \gamma_A(0) \).
Since \( A = \langle R; \mu_A, \gamma_A \rangle \) is an intuitionistic fuzzy right \( R \)-subgroup, it follows that
\[
\mu_A(x - y) \geq \min\{\mu_A(x), \mu_A(y)\} = \mu_A(0), \\
\gamma_A(x - y) \leq \max\{\gamma_A(x), \gamma_A(y)\} = \gamma_A(0)
\]
so that
\[
\mu_A(x - y) = \mu_A(0) \quad \text{and} \quad \gamma_A(x - y) = \gamma_A(0).
\]
Thus \( x - y \in R_A \). For any \( x \in R_A \) and \( r \in R \), we have
\[
\mu_A(xr) \geq \mu_A(x) = \mu_A(0), \quad \gamma_A(xr) \leq \gamma_A(x) = \gamma_A(0).
\]
Hence \( xr \in R_A \), and consequently \( R_A \) is a right \( R \)-subgroup of \( R \).

Let \( A = \langle R; \mu_A, \gamma_A \rangle \) be an IFS in a set \( R \) and let \( \alpha, \beta \in [0,1] \) be such that \( \alpha + \beta \leq 1 \). Then the set
\[
R_A^{(\alpha,\beta)} := \{ x \in R \mid \mu_A(x) \geq \alpha, \gamma_A(x) \leq \beta \}
\]
is called an \((\alpha,\beta)\)-level subset of \( A = \langle R; \mu_A, \gamma_A \rangle \).

Theorem 3.3. Let \( A = \langle R; \mu_A, \gamma_A \rangle \) be an intuitionistic fuzzy right \( R \)-subgroup of a near-ring \( R \). Then \( R_A^{(\alpha,\beta)} \) is a right \( R \)-subgroup of \( R \) for every \((\alpha,\beta) \in \text{Im}(\mu_A) \times \text{Im}(\gamma_A) \) with \( \alpha + \beta \leq 1 \).

Proof. Let \( x, y \in R_A^{(\alpha,\beta)} \). Then \( \mu_A(x) \geq \alpha, \gamma_A(x) \leq \beta, \mu_A(y) \geq \alpha, \) and \( \gamma_A(y) \leq \beta \) which imply that
\[
\mu_A(x - y) \geq \min\{\mu_A(x), \mu_A(y)\} \geq \alpha, \\
\gamma_A(x - y) \leq \max\{\gamma_A(x), \gamma_A(y)\} \leq \beta.
\]
Thus \( x - y \in R_A^{(\alpha,\beta)} \). Let \( r \in R \) and \( x \in R_A^{(\alpha,\beta)} \). Then
\[
\mu_A(xr) \geq \mu_A(x) \geq \alpha \quad \text{and} \quad \gamma_A(xr) \leq \gamma_A(x) \leq \beta;
\]
hence \( xr \in R_A^{(\alpha,\beta)} \). Therefore \( R_A^{(\alpha,\beta)} \) is a right \( R \)-subgroup of \( R \).
Theorem 3.4. Let $A = \langle R; \mu_A, \gamma_A \rangle$ be an IFS in a near-ring $R$ such that $R_A^{(\alpha, \beta)}$ is a right $R$-subgroup of $R$ for every $(\alpha, \beta) \in \text{Im}(\mu_A) \times \text{Im}(\gamma_A)$ with $\alpha + \beta \leq 1$. Then $A = \langle R; \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy right $R$-subgroup of $R$.

Proof. Let $x, y \in R$ and let $A(x) = (\alpha_1, \beta_1)$ and $A(y) = (\alpha_2, \beta_2)$, i.e.,

$$
\mu_A(x) = \alpha_1, \quad \gamma_A(x) = \beta_1, \quad \mu_A(y) = \alpha_2, \quad \gamma_A(y) = \beta_2.
$$

Then $x \in R_A^{(\alpha_1, \beta_1)}$ and $y \in R_A^{(\alpha_2, \beta_2)}$. We may assume that $(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2)$, i.e., $\alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$ without loss of generality. It follows that $R_A^{(\alpha_2, \beta_2)} \subseteq R_A^{(\alpha_1, \beta_1)}$ so that $x, y \in R_A^{(\alpha_1, \beta_1)}$. Since $R_A^{(\alpha_1, \beta_1)}$ is a right $R$-subgroup of $R$, we have $x - y \in R_A^{(\alpha_1, \beta_1)}$ and $xr \in R_A^{(\alpha_1, \beta_1)}$ for all $r \in R$. Thus

$$
\mu_A(x - y) \geq \alpha_1 = \min\{\alpha_1, \alpha_2\} = \min\{\mu_A(x), \mu_A(y)\},
\gamma_A(x - y) \leq \beta_1 = \max\{\beta_1, \beta_2\} = \max\{\gamma_A(x), \gamma_A(y)\},
\mu_A(xr) \geq \alpha_1 = \min\{\mu_A(x), \mu_A(y)\},
\gamma_A(xr) \leq \beta_1 = \max\{\gamma_A(x), \gamma_A(y)\}.
$$

Consequently, $A = \langle R; \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy right $R$-subgroup of $R$. \hfill \square

Note that

$$
R_A^{(\alpha, \beta)} = \left\{ x \in R \mid \mu_A(x) \geq \alpha, \quad \gamma_A(x) \leq \beta \right\}
= \left\{ x \in R \mid \mu_A(x) \geq \alpha \right\} \cap \left\{ x \in R \mid \gamma_A(x) \leq \beta \right\}
= U(\mu_A; \alpha) \cap L(\gamma_A; \beta).
$$

Hence we have the following corollary.

Corollary 3.5. Let $A = \langle R; \mu_A, \gamma_A \rangle$ be an IFS in a near-ring $R$. Then $A = \langle R; \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy right $R$-subgroup of $R$ if and only if $U(\mu_A; \alpha)$ and $L(\gamma_A; \beta)$ are right $R$-subgroups of $R$ for every $\alpha \in [0, \mu_A(0)]$ and $\beta \in [\gamma_A(0), 1]$ with $\alpha + \beta \leq 1$.

Corollary 3.6. Let $I$ be a right $R$-subgroup of a near-ring $R$ and let $A = \langle R; \mu_A, \gamma_A \rangle$ be an IFS in $R$ defined by

$$
\mu_A(x) := \begin{cases} 
\alpha_0 & \text{if } x \in I, \\
\alpha_1 & \text{otherwise},
\end{cases}
\gamma_A(x) := \begin{cases} 
\beta_0 & \text{if } x \in I, \\
\beta_1 & \text{otherwise},
\end{cases}
$$

for all $x \in R$ where $0 \leq \alpha_1 < \alpha_0$, $0 \leq \beta_0 < \beta_1$, and $\alpha_i + \beta_i \leq 1$ for $i = 1, 2$. Then $A = \langle R; \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy right $R$-subgroup of $R$. 


Proposition 3.7. Let $A = \langle R; \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy right $R$-subgroup of a near-ring $R$ and

$$(\alpha_1, \beta_1), \ (\alpha_2, \beta_2) \in \text{Im}(\mu_A) \times \text{Im}(\gamma_A)$$

with $\alpha_i + \beta_i \leq 1$ for $i = 1, 2$. Then $R_A^{(\alpha_1, \beta_1)} = R_A^{(\alpha_2, \beta_2)}$ if and only if $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$.

Proof. If $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$, then clearly $R_A^{(\alpha_1, \beta_1)} = R_A^{(\alpha_2, \beta_2)}$. Assume that $R_A^{(\alpha_1, \beta_1)} \neq R_A^{(\alpha_2, \beta_2)}$. Since $(\alpha_1, \beta_1) \in \text{Im}(\mu_A) \times \text{Im}(\gamma_A)$, there exists $x \in R$ such that $\mu_A(x) = \alpha_1$ and $\gamma_A(x) = \beta_1$. It follows that $x \in R_A^{(\alpha_1, \beta_1)} = R_A^{(\alpha_2, \beta_2)}$ so that $\alpha_1 = \mu_A(x) \geq \alpha_2$ and $\beta_1 = \gamma_A(x) \leq \beta_2$.

Similarly we have $\alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$. Hence $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$. \qed

Theorem 3.8. Consider a chain of right $R$-subgroups of a near-ring $R$

$G_0 \subset G_1 \subset \cdots \subset G_n = R$,

where $\subset$ denotes proper inclusion. Then there exists an intuitionistic fuzzy right $R$-subgroup of $R$ whose upper and lower level right $R$-subgroups are exactly the right $R$-subgroups in the above chain.

Proof. Let $\{\alpha_k \mid k = 0, 1, \cdots, n\}$ (resp. $\{\beta_k \mid k = 0, 1, \cdots, n\}$) be a finite decreasing (resp. increasing) sequence in $[0, 1]$ such that $\alpha_i + \beta_i \leq 1$ for $i = 0, 1, \cdots, n$. Let $A = \langle R; \mu_A, \gamma_A \rangle$ be an IFS in $R$ defined by

$$\mu_A(G_0) = \alpha_0, \ \gamma_A(G_0) = \beta_0, \ \mu_A(G_k \setminus G_{k-1}) = \alpha_k$$

and

$$\gamma_A(G_k \setminus G_{k-1}) = \beta_k \text{ for } 0 < k \leq n.$$ 

Let $x, y \in R$. If $x, y \in G_k \setminus G_{k-1}$, then

$$x - y \in G_k, \ \mu_A(x) = \alpha_k = \mu_A(y)$$

and

$$\gamma_A(x) = \beta_k = \gamma_A(y).$$

It follows that

$$\mu_A(x - y) \geq \alpha_k = \min\{\mu_A(x), \mu_A(y)\}$$

and

$$\gamma_A(x - y) \leq \beta_k = \max\{\gamma_A(x), \gamma_A(y)\}.$$ 

For $i > j$, if $x \in G_i \setminus G_{i-1}$ and $y \in G_j \setminus G_{j-1}$, then $\mu_A(x) = \alpha_i < \alpha_j = \mu_A(y)$, $\gamma_A(x) = \beta_i > \beta_j = \gamma_A(y)$ and $xy \in G_i$. Hence

$$\mu_A(x - y) \geq \alpha_i = \min\{\mu_A(x), \mu_A(y)\}$$
and
\[ \gamma_A(x - y) \leq \beta_i = \max\{\gamma_A(x), \gamma_A(y)\}. \]
Now let \( x \in R \). Then there exists \( k \in \{0, 1, \ldots, n\} \) such that \( x \in G_k \setminus G_{k-1} \).
Since \( G_k \) is a right \( R \)-subgroup, we have \( xr \in G_k \) for all \( r \in R \). It follows that
\[ \mu_A(xr) \geq \alpha_k = \mu_A(x), \quad \gamma_A(xr) \leq \beta_k = \gamma_A(x). \]
Therefore \( A = \langle R; \mu_A, \gamma_A \rangle \) is an intuitionistic fuzzy right \( R \)-subgroup of \( R \).
Note that \( \text{Im}(\mu_A) = \{\alpha_0, \alpha_1, \ldots, \alpha_n\} \) and \( \text{Im}(\gamma_A) = \{0, \beta_1, \ldots, \beta_n\} \). It follows that the upper level right \( R \)-subgroups and the lower level right \( R \)-subgroups of \( A = \langle R; \mu_A, \gamma_A \rangle \) are given by the chain of right \( R \)-subgroups
\[ U(\mu_A; \alpha_0) \subset U(\mu_A; \alpha_1) \subset \cdots \subset U(\mu_A; \alpha_n) = R \]
and
\[ L(\gamma_A; \beta_0) \subset L(\gamma_A; \beta_1) \subset \cdots \subset L(\gamma_A; \beta_n) = R, \]
respectively. Obviously, we have
\[ U(\mu_A; \alpha_0) = \{x \in R \mid \mu_A(x) \geq \alpha_0\} = G_0, \]
\[ L(\gamma_A; \beta_0) = \{x \in R \mid \gamma_A(x) \leq \beta_0\} = G_0. \]
We now prove that
\[ U(\mu_A; \alpha_k) = G_k = L(\gamma_A; \beta_k) \quad \text{for} \quad 0 < k \leq n. \]
Clearly \( G_k \subseteq U(\mu_A; \alpha_k) \) and \( G_k \subseteq L(\gamma_A; \beta_k) \). If \( x \in U(\mu_A; \alpha_k) \), then \( \mu_A(x) \geq \alpha_k \) and so \( x \notin G_i \) for \( i > k \). Hence
\[ \mu_A(x) \in \{\alpha_1, \alpha_2, \ldots, \alpha_k\}, \]
which implies \( x \in G_j \) for some \( j \leq k \). Since \( G_j \subseteq G_k \), it follows that \( x \in G_k \).
Consequently,
\[ U(\mu_A; \alpha_k) = G_k \quad \text{for} \quad 0 \leq k \leq n. \]
Now if \( y \in L(\gamma_A; \beta_k) \), then \( \gamma_A(y) \leq \beta_k \) and thus \( y \notin G_i \) for \( i > k \). Hence
\[ \gamma_A(y) \in \{\beta_1, \beta_2, \ldots, \beta_k\} \]
and so \( y \in G_j \) for some \( j \leq k \). Since \( G_j \subseteq G_k \), we have \( y \in G_k \). Therefore
\[ L(\gamma_A; \beta_k) = G_k \quad \text{for} \quad 0 \leq k \leq n. \]
This completes the proof. \( \square \)

**Theorem 3.9.** Let \( \left\{ G_\alpha \mid \alpha \in \Lambda \subseteq [0, \frac{1}{2}] \right\} \) be a finite collection of right \( R \)-subgroups of a near-ring \( R \) such that \( R = \bigcup_{\alpha \in \Lambda} G_\alpha \), and for every \( \alpha, \beta \in \Lambda \), \( \alpha < \beta \) if and only if \( G_\beta \subseteq G_\alpha \). Then an IFS \( A = \langle R; \mu_A, \gamma_A \rangle \) in \( R \) defined by
\[ \mu_A(x) = \sup\{\alpha \in \Lambda \mid x \in G_\alpha\} \quad \text{and} \quad \gamma_A(x) = \inf\{\alpha \in \Lambda \mid x \in G_\alpha\} \]
for all \( x \in R \) is an intuitionistic fuzzy right \( R \)-subgroup of \( R \).
Proof. According to Corollary 3.5, it is sufficient to show that the nonempty sets $U(\mu_A; \alpha)$ and $L(\gamma_A; \beta)$ are right $R$-subgroups of $R$ for every $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$. In order to show that $U(\mu_A; \alpha)$ is a right $R$-subgroup, we divide into the following two cases:

(i) $\alpha = \sup\{\delta \in \Lambda \mid \delta < \alpha\}$

and

(ii) $\alpha \neq \sup\{\delta \in \Lambda \mid \delta < \alpha\}$.

Case (i) implies that

$$x \in U(\mu_A; \alpha) \iff x \in G_\delta \text{ for all } \delta < \alpha$$

$$\iff x \in \bigcap_{\delta < \alpha} G_\delta,$$

so that $U(\mu_A; \alpha) = \bigcap_{\delta < \alpha} G_\delta$, which is a right $R$-subgroup of $R$.

For the case (ii), we claim that

$$U(\mu_A; \alpha) = \bigcup_{\delta \geq \alpha} G_\delta.$$ 

If $x \in \bigcup_{\delta \geq \alpha} G_\delta$, then $x \in G_\delta$ for some $\delta \geq \alpha$. It follows that $\mu_A(x) \geq \delta \geq \alpha$ so that $x \in U(\mu_A; \alpha)$. This proves that

$$\bigcup_{\delta \geq \alpha} G_\delta \subset U(\mu_A; \alpha).$$

Now assume that $x \notin \bigcup_{\delta \geq \alpha} G_\delta$. Then $x \notin G_\delta$ for all $\delta \geq \alpha$. Since $\alpha \neq \sup\{\delta \in \Lambda \mid \delta < \alpha\}$, there exists $\varepsilon > 0$ such that $(\alpha - \varepsilon, \alpha) \cap \Lambda = \emptyset$. Hence $x \notin G_\delta$ for all $\delta > \alpha - \varepsilon$, which means that if $x \in G_\delta$ then $\delta \leq \alpha - \varepsilon$. Thus $\mu_A(x) \leq \alpha - \varepsilon < \alpha$, and so $x \notin U(\mu_A; \alpha)$. Therefore

$$U(\mu_A; \alpha) = \bigcup_{\delta \geq \alpha} G_\delta.$$ 

Next we show that $L(\gamma_A; \beta)$ is a right $R$-subgroup of $R$ for all $\beta \in [\gamma_A(0), 1]$. We consider the following two cases:

(iii) $\beta = \inf\{\delta \in \Lambda \mid \beta < \delta\}$

and

(iv) $\beta \neq \inf\{\delta \in \Lambda \mid \beta < \delta\}$.

For the case (iii) we have

$$x \in L(\gamma_A; \beta) \iff x \in G_\delta \text{ for all } \beta < \delta$$

$$\iff x \in \bigcap_{\beta < \delta} G_\delta,$$

and hence $L(\gamma_A; \beta) = \bigcap_{\beta < \delta} G_\delta$, which is a right $R$-subgroup of $R$. 
For the case (iv), we will show that
\[ L(\gamma_A; \beta) = \bigcup_{\beta \geq \delta} G_{\delta}. \]
If \( x \in \bigcup_{\beta \geq \delta} G_{\delta} \), then \( x \in G_{\delta} \) for some \( \beta \geq \delta \). It follows that \( \gamma_A(x) \leq \delta \leq \beta \) so that \( x \in L(\gamma_A; \beta) \). Hence
\[ \bigcup_{\beta \geq \delta} G_{\delta} \subset L(\gamma_A; \beta). \]
Conversely, if \( x \notin \bigcup_{\beta \geq \delta} G_{\delta} \), then \( x \notin G_{\delta} \) for all \( \delta \leq \beta \). Since \( \beta \neq \inf \{ \delta \in \Lambda \mid \beta < \delta \} \), there exists \( \varepsilon > 0 \) such that \( (\beta, \beta + \varepsilon) \cap \Lambda = \emptyset \), which implies that \( x \notin G_{\delta} \) for all \( \delta < \beta + \varepsilon \), that is, if \( x \in G_{\delta} \), then \( \delta \geq \beta + \varepsilon \). Thus \( \gamma_A(x) \geq \beta + \varepsilon > \beta \), that is, \( x \notin L(\gamma_A; \beta) \). Therefore
\[ L(\gamma_A; \beta) \subset \bigcup_{\beta \geq \delta} G_{\delta} \]
and consequently \( L(\gamma_A; \beta) = \bigcup_{\beta \geq \delta} G_{\delta} \). This completes the proof. \( \square \)

**Definition 3.10.** An intuitionistic fuzzy right \( R \)-subgroup \( A = \langle R; \mu_A, \gamma_A \rangle \) of a near-ring \( R \) is said to be normal if there exists \( x \in R \) such that \( \mu_A(x) = 1 \) and \( \gamma_A(x) = 0 \).

Note that if an intuitionistic fuzzy right \( R \)-subgroup \( A = \langle R; \mu_A, \gamma_A \rangle \) of a near-ring \( R \) is normal, then \( \mu_A(0) = 1 \) and \( \gamma_A(0) = 0 \); hence \( A = \langle R; \mu_A, \gamma_A \rangle \) is a normal intuitionistic fuzzy right \( R \)-subgroup of \( R \) if and only if \( \mu_A(0) = 1 \) and \( \gamma_A(0) = 0 \).

**Example 3.11.** (1) Let \( R = \{a, b, c, d\} \) be a set with two binary operations as follows:

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<tr>
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<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
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<tbody>
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Then \( (R, +, \cdot) \) is a near-ring. We define an IFS \( A = \langle R; \mu_A, \gamma_A \rangle \) in \( R \) by
\[ A = \langle R; \left( \begin{array}{cccc}
\alpha & \beta & \gamma & \delta \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{array} \right), \left( \begin{array}{cccc}
\alpha & \beta & \gamma & \delta \\
\frac{3}{5} & \frac{3}{5} & \frac{3}{5} & \frac{3}{5}
\end{array} \right) \rangle. \]
Then $A = \langle R; \mu_A, \gamma_A \rangle$ is a normal intuitionistic fuzzy subgroup of $(R, +)$, and we have that

\[
\mu_A(xr) \geq \mu_A(x) \quad \text{and} \quad \gamma_A(xr) \leq \gamma_A(x) \quad \text{for all} \quad r, x \in R.
\]

Hence $A = \langle R; \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy right $R$-subgroup of $R$.

(2) Let $R = \{a, b, c, d\}$ be a set with two binary operations as follows:

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<th>a</th>
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Then $(R, +, \cdot)$ is a near-ring. We define an IFS $B = \langle R; \mu_B, \gamma_B \rangle$ in $R$ by

\[
B = \langle R; \left(\begin{array}{cccc}
a & b & c & d \\
a & a & a & a \\
b & a & a & a \\
c & a & a & a \\
d & a & a & a \\
\end{array}\right)\rangle.
\]

Then $B = \langle R; \mu_B, \gamma_B \rangle$ is a normal intuitionistic fuzzy subgroup of $(R, +)$, and we have that

\[
\mu_B(xr) \geq \mu_B(x) \quad \text{and} \quad \gamma_B(xr) \leq \gamma_B(x) \quad \text{for all} \quad r, x \in R.
\]

Hence $B = \langle R; \mu_B, \gamma_B \rangle$ is an intuitionistic fuzzy right $R$-subgroup of $R$.

Note that every intuitionistic fuzzy $R$-subgroup of a near-ring $R$ is an intuitionistic fuzzy subnear-ring of $R$ (see [6]). But the converse is not true in general as seen in the following example.

**Example 3.12.** (1) Let $R = \{a, b, c, d\}$ be a set with two binary operations as follows:

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Then $(R, +, \cdot)$ is a near-ring. We define an IFS $A = \langle R; \mu_A, \gamma_A \rangle$ in $R$ by

\[
A = \langle R; \left(\begin{array}{cccc}
a & b & c & d \\
a & a & a & a \\
b & a & a & a \\
c & a & a & a \\
d & a & a & a \\
\end{array}\right)\rangle.
\]

Then $A = \langle R; \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy subnear-ring of $R$. But $A = \langle R; \mu_A, \gamma_A \rangle$ is not an intuitionistic fuzzy right $R$-subgroup of $R$ since

\[
\mu_A(b \cdot c) = \mu_A(c) = 0.4 < 0.5 = \mu_A(b)
\]
and/or
\[ \gamma_A(b \cdot c) = \gamma_A(c) = 0.5 > 0.3 = \gamma_A(b). \]

(2) Let \( \mathbb{R} \) be a ring of real numbers with the usual addition “+” and multiplication “⋅”. Then \( (\mathbb{R}, +, \cdot) \) is a near-ring. Let \( B = (\mathbb{R}; \mu_B, \gamma_B) \) be an IFS in \( \mathbb{R} \) defined by

\[
\mu_B(r) := \begin{cases} 
1 & \text{if } r = 0, \\
\frac{1}{2} & \text{if } r \in \mathbb{Z} \setminus \{0\}, \\
\frac{1}{4} & \text{if } r \in \mathbb{Q} \setminus \mathbb{Z}, \\
0 & \text{if } r \in \mathbb{R} \setminus \mathbb{Q}, 
\end{cases}
\]

\[
\gamma_B(r) := \begin{cases} 
0 & \text{if } r = 0, \\
\frac{1}{4} & \text{if } r \in \mathbb{Z} \setminus \{0\}, \\
\frac{1}{2} & \text{if } r \in \mathbb{Q} \setminus \mathbb{Z}, \\
1 & \text{if } r \in \mathbb{R} \setminus \mathbb{Q}, 
\end{cases}
\]

for \( r \in \mathbb{R} \) where \( \mathbb{Q} \) and \( \mathbb{Z} \) are rings of the rational numbers and the integers, respectively. Then \( B = (\mathbb{R}; \mu_B, \gamma_B) \) is an intuitionistic fuzzy subnear-ring of \( \mathbb{R} \). But

\[
\mu_B(2 \cdot \frac{1}{3}) = \mu_B(\frac{2}{3}) = \frac{1}{4} < \frac{1}{2} = \mu_B(2)
\]

and/or

\[
\gamma_B(2 \cdot \frac{1}{3}) = \gamma_B(\frac{2}{3}) = \frac{1}{2} > \frac{1}{4} = \gamma_B(2).
\]

Hence \( B = (\mathbb{R}; \mu_B, \gamma_B) \) is not an intuitionistic fuzzy right \( \mathbb{R} \)-subgroup of \( \mathbb{R} \).

**Theorem 3.13.** Let \( A = (\mathbb{R}; \mu_A, \gamma_A) \) be an intuitionistic fuzzy right \( \mathbb{R} \)-subgroup of a near-ring \( R \) and let \( \mu_{A+} \) and \( \gamma_{A+} \) be fuzzy sets in \( R \) defined by

\[ \mu_{A+}(x) = \mu_A(x) + 1 - \mu_A(0), \quad \gamma_{A+}(x) = \gamma_A(x) - \gamma_A(0) \]

for all \( x \in R \) respectively. If

\[ \mu_{A+}(x) + \gamma_{A+}(x) \leq 1 \]

for all \( x \in R \), then

\[ A^+ = (\mathbb{R}; \mu_{A+}, \gamma_{A+}) \]

is a normal intuitionistic fuzzy right \( \mathbb{R} \)-subgroup of \( R \) containing \( A = (\mathbb{R}; \mu_A, \gamma_A) \).

**Proof.** Assume that \( \mu_{A+}(x) + \gamma_{A+}(x) \leq 1 \) for all \( x \in R \). Then \( A^+ = (\mathbb{R}; \mu_{A+}, \gamma_{A+}) \) is an IFS in \( R \). Let \( x, y \in R \). Then

\[
\min \left\{ \mu_{A+}(x), \mu_{A+}(y) \right\} = \min \left\{ \mu_A(x) + 1 - \mu_A(0), \mu_A(y) + 1 - \mu_A(0) \right\} 
\]

\[ = \min \left\{ \mu_A(x), \mu_A(y) \right\} + 1 - \mu_A(0) \]

\[ \leq \mu_A(x - y) + 1 - \mu_A(0) \]

\[ = \mu_{A+}(x - y), \]
max \left\{ \gamma_A^+(x), \gamma_A^+(y) \right\} = \max \left\{ \gamma_A(x) - \gamma_A(0), \gamma_A(y) - \gamma_A(0) \right\} \\
= \max \left\{ \gamma_A(x), \gamma_A(y) \right\} - \gamma_A(0) \\
\geq \gamma_A(x - y) - \gamma_A(0) \\
= \gamma_A^+(x - y),

and for all \( x, r \in R \), we have

\[
\begin{align*}
\mu_{A^+}(xr) &= \mu_A(xr) + 1 - \mu_A(0) \\
&\geq \mu_A(x) + 1 - \mu_A(0) \\
&= \mu_{A^+}(x) \\
\gamma_{A^+}(xr) &= \gamma_A(xr) - \gamma_A(0) \\
&\leq \gamma_A(x) - \gamma_A(0) \\
&= \gamma_{A^+}(x).
\end{align*}
\]

Hence \( A^+ = (R; \mu_{A^+}, \gamma_{A^+}) \) is an intuitionistic fuzzy right \( R \)-subgroup of \( R \).

Clearly \( \mu_{A^+}(0) = 1, \ \gamma_{A^+}(0) = 0, \ \mu_A \subset \mu_{A^+} \) and \( \gamma_A \supset \gamma_{A^+} \).

This completes the proof. \( \Box \)

**Corollary 3.14.** Let \( \mu_{A^+}, \gamma_{A^+} \) and \( A = (R; \mu_A, \gamma_A) \) be as in Theorem 3.13 such that

\[
\mu_{A^+}(x) + \gamma_{A^+}(x) \leq 1 \text{ for all } x \in R.
\]

If \( \mu_{A^+}(x) = 0 \) and \( \gamma_{A^+}(x) = 1 \) for some \( x \in R \), then \( \mu_A(x) = 0 \) and \( \gamma_A(x) = 1 \).

**Proof.** Straightforward. \( \Box \)

**Theorem 3.15.** Let \( A = (R; \mu_A, \gamma_A) \) and \( B = (R; \mu_B, \gamma_B) \) be intuitionistic fuzzy right \( R \)-subgroups of a near-ring \( R \). If \( A \subset B \), that is, \( \mu_A \subset \mu_B \) and \( \gamma_A \supset \gamma_B \), and \( A(0) = B(0) \), that is, \( \mu_A(0) = \mu_B(0) \) and \( \gamma_A(0) = \gamma_B(0) \), then \( R_A \subset R_B \).

**Proof.** Assume that \( A \subset B \) and \( A(0) = B(0) \). If \( x \in R_A \), then

\[
(\forall x \in R) \left( \mu_B(x) \geq \mu_A(x) = \mu_A(0) = \mu_B(0) \right).
\]

\[
(\forall x \in R) \left( \gamma_B(x) \leq \gamma_A(x) = \gamma_A(0) = \gamma_B(0) \right).
\]

Since \( \mu_B(0) \geq \mu_B(x) \) and \( \gamma_B(x) \geq \gamma_B(0) \) for all \( x \in R \), it follows that \( \mu_B(x) = \mu_B(0) \) and \( \gamma_B(x) = \gamma_B(0) \) so that \( x \in R_B \). This completes the proof. \( \Box \)
Corollary 3.16. If \( A = \langle R; \mu_A, \gamma_A \rangle \) and \( B = \langle R; \mu_B, \gamma_B \rangle \) are normal intuitionistic fuzzy right \( R \)-subgroups of a near-ring \( R \) satisfying \( A \subset B \), then \( R_A \subset R_B \).

4. Conclusions

We gave a characterization of an intuitionistic fuzzy right \( R \)-subgroup in a near-ring \( R \), and investigated some properties. We established an intuitionistic fuzzy right \( R \)-subgroup by using a collection of right \( R \)-subgroups and a chain of right \( R \)-subgroups, respectively. We considered the notion of a normal intuitionistic fuzzy right \( R \)-subgroup.

Future research will focus on studying the intuitionistic fuzzification of a right \( R \)-subgroup in a near-ring \( R \) by using a triangular norm, on studying the Cartesian product of intuitionistic fuzzy right \( R \)-subgroups, and on considering intuitionistic fuzzy (congruence) relations on near-rings.

References


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