A CENTRAL LIMIT THEOREM FOR GENERAL WEIGHTED SUM OF LNQD RANDOM VARIABLES AND ITS APPLICATION

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Abstract. In this paper we derive the central limit theorem for \( \sum_{i=1}^{n} a_{ni} \xi_i \), where \( \{a_{ni}, 1 \leq i \leq n\} \) is a triangular array of non-negative numbers such that \( \sup_n \sum_{i=1}^{n} a_{ni}^2 < \infty \), \( \max_{1 \leq i \leq n} a_{ni} \to 0 \) as \( n \to \infty \) and \( \xi_i \)s are a linearly negative quadrant dependent sequence. We also apply this result to consider a central limit theorem for a partial sum of a generalized linear process \( X_n = \sum_{j=-\infty}^{\infty} a_{k+j} \xi_j \).

1. Introduction and results

Lehmann[6] introduced a simple and natural definition of positive (negative) dependence: A sequence \( \{\xi_i, i \in \mathbb{Z}\} \) of random variables is said to be pairwise positive (negative) quadrant dependent (pairwise PQD(NQD)) if for any real \( \alpha_i, \alpha_j \) and \( i \neq j \), \( P(\xi_i > \alpha_i, \xi_j > \alpha_j) \geq (\leq) P(\xi_i > \alpha_i)P(\xi_j > \alpha_j) \). A concept stronger than PQD(NQD) was introduced by Newman[7]: A sequence \( \{\xi_i, i \in \mathbb{Z}\} \) of random variables is said to be linearly positive(negative) quadrant dependent(LPQD(LNQD)) if for every pair of disjoint subsets \( A, B \subset \mathbb{Z} \) and positive \( r_i \)'s

\[
\sum_{i \in A} r_i \xi_i \quad \text{and} \quad \sum_{j \in B} r_j \xi_j \quad \text{are PQD (NQD)}.
\]

Newman[7] established the central limit theorem for a strictly stationary LPQD(or LNQD) process and Birkel[2] also obtained a functional central limit theorem for LPQD process which is used to obtain the functional central limit theorem for LNQD process. Kim and

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Baek[5] extended this result to a stationary linear process of the form
\[ X_k = \sum_{j=0}^{\infty} a_j \xi_{k-j}, \]
where \( \{a_j\} \) is a sequence of real numbers with \( \sum_{j=0}^{\infty} |a_j| < \infty \) and \( \{\xi_k\} \) is a strictly stationary LPQD process with \( E\xi_i = 0, \; 0 < E\xi_i^2 < \infty \), which can be extended to the LNQD case by similar method.

In this paper, we derive a central limit theorem for a linearly negative quadrant dependent sequence in a double array, weakening the strictly stationarity assumption with uniform integrability (see Theorem 1.1 below) and apply this result to obtain a central limit theorem for a partial sum of a linear process \( X_n = \sum_{j=-\infty}^{\infty} a_{k+j} \xi_j \) generated by linearly negative quadrant dependent sequence \( \{\xi_j\} \) (see Theorem 1.2 below).

**Theorem 1.1.** Let \( \{a_{ni}, \; 1 \leq i \leq n\} \) be a triangular array of non-negative numbers such that

\begin{equation}
\sup_n \sum_{i=1}^{n} a_{ni}^2 < \infty
\end{equation}

and

\begin{equation}
\max_{1 \leq i \leq n} a_{ni} \to 0 \text{ as } n \to \infty.
\end{equation}

Let \( \{\xi_i\} \) be a centered sequence of linearly negative quadrant dependent random variables such that

\begin{itemize}
  \item \( \{\xi_i^2\} \) is an uniformly integrable family,
  \item \( \text{Var}(\sum_{i=1}^{n} a_{ni} \xi_i) = 1, \)
\end{itemize}

and

\begin{equation}
\sum_{j:|i-j| \geq u} \text{Cov}(\xi_i, \xi_j) \to 0 \text{ as } u \to \infty \text{ uniformly in } i \geq 1
\end{equation}

(see Cox and Grimmet[3]). Then

\[ \sum_{i=1}^{n} a_{ni} \xi_i \xrightarrow{D} N(0,1) \text{ as } n \to \infty. \]

**Remark.** Theorem 1.1 extends the Newman’s[7] central limit theorem for strictly stationary LNQD sequence from equal weights to general weights, weakening at the same time the assumption of stationarity.
COROLLARY 1.1. Let \( \{\xi_i\} \) be a centered sequence of linearly negative quadrant dependent random variables such that \( \{\xi_i^2\} \) is a uniformly integrable family and let \( \{a_{ni}, 1 \leq i \leq n\} \) be a triangular array of nonnegative numbers such that

\[
\sup_n \sum_{i=1}^{n} \frac{a_{ni}^2}{\sigma_n^2} < \infty,
\]

\[
\max_{1 \leq i \leq n} \frac{a_{ni}}{\sigma_n} \to 0 \text{ as } n \to \infty,
\]

where \( \sigma_n^2 = \text{Var}(\sum_{i=1}^{n} a_{ni} \xi_i) \). If (1.6) holds then, as \( n \to \infty \)

\[
\frac{1}{\sigma_n} \sum_{i=1}^{n} a_{ni} \xi_i \overset{D}{\to} N(0,1).
\]

THEOREM 1.2. Let \( \{a_j, j \in \mathbb{Z}\} \) be a sequence of nonnegative numbers such that \( \sum_j a_j < \infty \) and let \( \{\xi_j, j \in \mathbb{Z}\} \) be a sequence of linearly negative quadrant dependent random variables which is uniformly integrable in \( L_2 \) and satisfies (1.6). Let

\[
X_k = \sum_{j=-\infty}^{\infty} a_{k+j} \xi_j \text{ and } S_n = \sum_{i=1}^{n} X_i.
\]

Assume

\[
\inf_{n \geq 1} n^{-1} \sigma_n^2 > 0,
\]

where \( \sigma_n^2 = \text{Var}(S_n) \). Then

\[
\frac{S_n}{\sigma_n} \overset{D}{\to} N(0,1) \text{ as } n \to \infty.
\]

This result is an extension of Theorem 18.6.5 in Ibragimov and Linneike[4] from i.i.d. to the linearly negative quadrant dependence case by adding the condition (1.6) and improves on Kim and Baek's[5] result about central limit theorem for allinear processes generated by LNQD sequences.

2. Proofs

We starts with the following lemma.
Lemma 2.1. (Newman [8]) Let \( \{Z_i, 1 \leq i \leq n\} \) be a sequence of linearly negative quadrant dependent random variables with finite second moments. Then

\[
|E \exp(it \sum_{j=1}^{n} Z_j) - \prod_{j=1}^{n} E \exp(it Z_j)| \leq Ct^2 \left| \sum_{j=1}^{n} Z_j - \sum_{j=1}^{n} \text{Var}(Z_j) \right|
\]

for all \( t \in \mathbb{R} \), where \( C > 0 \) is an arbitrary constant, not depending on \( n \).

Proof of Theorem 1.1. Without loss of generality we assume that \( a_{ni} = 0 \) for all \( i > n \) and \( \sup_{n} E\xi_n^2 = M < \infty \). For every \( 1 \leq a < b \leq n \) and \( 1 \leq u \leq b - a \) we have, after a simple manipulations,

\[
0 \leq \sum_{i=a}^{b-u} a_{ni} \sum_{j=i+u}^{b} a_{nj} \text{Cov}(\xi_i, \xi_j) -
\]

\[
\leq \sup_k \left( \sum_{j:\mid k-j \mid \geq u} \text{Cov}(\xi_k, \xi_j)^{-} \right) \left( \sum_{i=a}^{b} a_{ni}^2 \right).
\]

In particular by definition of LNQD, we also have

\[
\text{Var} \left( \sum_{i=a}^{b} a_{ni} \xi_i \right) \leq M \sum_{i=a}^{b} a_{ni}^2.
\]

We shall construct now a triangular array of random variables \( \{Z_{ni}, 1 \leq i \leq n\} \) for which we shall make use of Lemma 2.1. Fix a small positive \( \epsilon \) and find a positive integer \( u = u_\epsilon \) such that, for every \( n \geq u + 1 \)

\[
0 \leq \sum_{i=1}^{n-u} a_{ni} \sum_{j=i+u}^{b} a_{nj} \text{Cov}(\xi_i, \xi_j) -
\]

\[
\leq \epsilon.
\]

This is possible because of (2.1) and (1.6). Denote by \([x]\) the integer part of \( x \) and define

\[
K = \left[ \frac{1}{\epsilon} \right]
\]

\[
Y_{nj} = \sum_{i=u(j+1)}^{u(j+1)} a_{ni} \xi_i, \quad j = 0, 1, \ldots,
\]

\[
A_j = \left\{ i : 2Kj \leq i < 2Kj + K, \text{Cov}(Y_{ni}, Y_{n,i+1})^{-} \leq \frac{2}{K} \sum_{i=2Kj}^{2Kj+K} \text{Var}(Y_{ni}) \right\}.
\]
Since \(2\text{Cov}(Y_{ni}, Y_{n,i+1}) \leq \text{Var}(Y_{ni}) + \text{Var}(Y_{n,i+1})\), we get that for every \(j\) the set \(A_j\) is not empty. Now we define the integers \(m_1, m_2, \cdots, m_n\) recurrently by \(m_0 = 0\):

\[m_{j+1} = \min\{m; m > m_j, m \in A_j\}\]

and define

\[Z_{nj} = \sum_{i=m_j+1}^{m_{j+1}} Y_{ni}, j = 0, 1, \cdots,\]

\[\Delta_j = \{u(m_j + 1) + 1, \cdots, u(m_{j+1} + 1)\}.

We observe that

\[Z_{nj} = \sum_{k \in \Delta_j} a_{nk} \xi_k, j = 0, 1, \cdots .\]

By definition of LNQD \(Z'_{nj}\)'s are linearly negative quadrant dependent, and from the fact that \(m_j \geq 2K(j - 1)\) and \(m_{j+1} \leq K(2j + 1)\) every set \(\Delta_j\) contains no more than \(3 Ku\) elements and \(m_{j+1}/m_j \to 1\) as \(j \to \infty\). Hence, for every fixed positive \(\epsilon\) by (1.2)–(1.5) the array \(\{Z_{nj} : i = 1, \cdots, n; n \geq 1\}\) satisfies the Lindeberg’s condition (see Stout [9]). It remains to observe that by Lemma 2.1 and construction.

\[|E \exp(it \sum_{j=1}^{n} Z_{nj}) - \prod_{j=1}^{n} E \exp(it Z_{nj})|\]

\[\leq ct^2 \{\text{Var}(\sum_{j=1}^{n} Z_{nj}) - \sum_{j=1}^{n} \text{Var}(Z_{nj})\}\]

\[\leq ct^2 \{2(\sum_{i=1}^{n} \text{Cov}(Z_{ni}, Z_{n,i+1})) \leq 2(\sum_{i=1}^{n} \sum_{j=i+2}^{n} \text{Cov}(Z_{ni}, Z_{nj}))\}

\[\leq ct^2 \{4 \sum_{i=1}^{n} a_{ni} \sum_{j=i+u}^{n} a_{nj} \text{Cov}(\xi_i, \xi_j) \leq 2 \sum_{j=1}^{n} \text{Cov}(Y_{n,m_j}, Y_{n,m_{j+1}})\}\]

\[\leq ct^2 \{4\epsilon + \frac{8}{K} \sum_{i=1}^{n} \text{Var}(Y_{ni})\}\]

\[= ct^2 \{4\epsilon + \frac{8}{K} \sum_{j=1}^{n} \text{Var}\left(\sum_{i=ui+1}^{u(j+1)} a_n \xi_i\right)\}\]
\[ \leq ct^2 \{ 4\epsilon + \frac{8M}{K} \sum_{j=1}^{n} \sum_{i=u_j+1} a_{n_i}^2 \} \]
\[ \leq c_1 t^2 \epsilon \{ 1 + \sup_n \sum_{i=1}^{n} a_{n_i}^2 \} \]
\[ \leq c_2 t^2 \epsilon. \]

Now the proof is complete by Theorem 4.2 in Billingsley[1].

PROOF OF COROLLARY 1.1. Let \( A_{ni} = \frac{a_{ni}}{\sigma_n} \). Then we have

\[ \max_{1 \leq i \leq n} A_{ni} \to 0 \text{ as } n \to \infty, \]
\[ \sup_n \sum_{i=1}^{n} A_{ni}^2 < \infty, \]
\[ \Var \left( \sum_{i=1}^{n} A_{ni} \xi_i \right) = 1. \]

Hence, by Theorem 1.1 the desired result (1.11) follows.

PROOF OF THEOREM 1.2. First note that \( \sum_j a_j^2 < \infty \) and without restricting the generality, we can assume \( \sup E \xi_k^2 = 1 \). Let

\[ S_n = \sum_{k=1}^{n} X_k = \sum_{j=-\infty}^{\infty} \left( \sum_{k=1}^{n} a_{k+j} \right) \xi_j. \]

In order to apply Theorem 1.1, we fix \( W_n \) such that \( \sum_{|j|>W_n} a_j^2 < n^{-3} \) and take \( k_n = W_n + n \). Then

\[ \frac{S_n}{\sigma_n} = \sum_{|j| \leq k_n} \left( \sum_{k=1}^{n} a_{k+j} \right) \frac{\xi_j}{\sigma_n} + \sum_{|j|>k_n} \left( \sum_{k=1}^{n} a_{k+j} \right) \frac{\xi_j}{\sigma_n} \]
\[ = T_n + U_n \text{ (say)}. \]

By Cauchy Schwarz inequality and assumptions we have the following estimate

\[ \Var(U_n) \leq \sum_{|j|>k_n} \Var \left( \sum_{k=1}^{n} a_{k+j} \frac{\xi_j}{\sigma_n} \right) \]
A central limit theorem for general weighted sum

\[ \leq \sum_{|j|>k_n} \left( \sum_{k=1}^{n} \frac{a_{k+j}}{\sigma_n} \right)^2 E\xi_j^2 \]

\[ \leq n\sigma_n^{-2} \sum_{|j|>k_n} \left( \sum_{k=1}^{n} a_{k+j}^2 \right) \]

\[ \leq n^2 \sigma_n^{-2} \sum_{|j|>k_n-n} a_j^2 \]

\[ \leq n^2 \sigma_n^{-2} \sum_{|j|>W_n} a_j^2 \]

\[ \leq n^{-1} \sigma_n^{-2} \to 0 \text{ as } n \to \infty, \]

which yields

(2.2) \[ U_n \to 0 \text{ in probability as } n \to \infty. \]

It remains only to prove that \( T_n \xrightarrow{D} N(0,1) \) by Theorem 4.1 of Billingsley[1]. Put

(2.3) \[ a_{nk} = \frac{\sum_{j=1}^{n} a_{k+j}}{\sigma_n}. \]

From assumption \( \sum_j a_j < \infty \) (\( a_j > 0 \)), (1.10) and (2.3) we obtain

\[ \sup_{-\infty<k<\infty} \sum_{j=1}^{n} a_{k+j} > 0 \text{ as } n \to \infty, \]

\[ \max_{1 \leq k \leq n} a_{nk} \to 0 \text{ as } n \to \infty, \]

\[ \sup_n \sum_{k=1}^{n} a_{nk}^2 < \infty. \]

Hence, by Theorem 1.1

(2.4) \[ T_n \xrightarrow{D} N(0,1) \]

and from (2.2) and (2.4) the desired result (1.10) follows. \qed

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