COUNTABLE RINGS WITH ACC ON ANNIHILATORS

YASUYUKI HIRANO AND HONG KEE KIM

Abstract. We consider countable rings with ascending chain condition on right annihilators. We determine the structure of a countable right p-injective Baer ring, a countable semiprime quasi-Baer ring and a countable quasi-Baer biregular ring.

1. Introduction

Rangaswamy[11, Theorem 1] proved that a countable regular Baer ring is a semisimple Artinian ring. Kim and Park[9] showed that in this theorem, the condition that “$R$ is countable” can be replaced by the condition that “$R$ has only countably many idempotents”. Recently, Birkenmeier, Kim and Park[2] studied rings with countably many direct summands. Recall that a ring is orthogonally finite if it has no infinite sets of nonzero orthogonal idempotents. A Baer ring is a ring in which the left (and right) annihilator of every subset is generated by an idempotent (see [12, Lemma 3.8.1, p.78]). From the proof of [11, Theorem 1] we know that a Baer ring satisfies the ascending chain condition on right ideals if and only if it is orthogonally finite. Rangaswamy’s proof consists of the following observation: (1) A countable Baer ring has only countably many right annihilators; (2) If a ring has only countably many right annihilators, it must be orthogonally finite; (3) An orthogonally finite von Neumann regular ring is a semisimple Artinian ring. In this paper, we refine the above observation, and then generalize Rangaswamy[11, Theorem 1]. First we prove some preliminary results. Next we consider p-injective rings. Clearly a von Neumann regular ring is a p-injective ring. We prove that a countable right p-injective Baer ring is a semisimple Artinian ring. Finally we consider semiprime quasi-Baer
rings. We prove that a countable semiprime quasi-Baer ring is a finite direct sum of prime rings and that a countable quasi-Baer biregular ring is a finite direct sum of simple rings.

2. Preliminary results

For a subset $X$ of a ring $R$, $r(X)$ (resp. $l(X)$) denote the right (resp. left) annihilator of $X$ in $R$. First we consider the relationship between the orthogonally finiteness, the ascending chain condition (ACC) on right annihilators and the descending chain condition (DCC) on right annihilators.

**Theorem 2.1.** Let $R$ be a ring. Then the following statements are equivalent:

1. $R$ satisfies ACC (resp. DCC) on right annihilators;
2. $R$ is orthogonally finite and $R$ satisfies ACC (resp. DCC) on right annihilators containing no nonzero idempotents.

**Proof.** (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). Assume that $R$ is orthogonally finite and $R$ satisfies ACC on right annihilators containing no nonzero idempotents. Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of right annihilators. Since $R$ is orthogonally finite, $I = \bigcup I_i$ contains no infinite set of orthogonal idempotents. Hence there exists an idempotent $e \in I$ such that $(1-e)I$ contains no nonzero idempotent. Then $e \in I_j$ for some $j$. By hypothesis on $e$, $(1-e)I_j \subseteq (1-e)I_{j+1} \subseteq \cdots$ is an ascending chain of right annihilators containing no nonzero idempotents. Then there exists $k \geq j$ such that $(1-e)I_k = (1-e)I_{k+1} = \cdots$. Since $e \in I_j$, $I_m = eR \oplus (1-e)I_m$ for all $m \geq j$. Therefore $I_k = eR \oplus (1-e)I_k = eR \oplus (1-e)I_{k+m} = I_{k+m}$ for all $m \geq 0$.

Next assume that $R$ is orthogonally finite and $R$ satisfies DCC on right annihilators containing no nonzero idempotents. Let $I_1 \supseteq I_2 \supseteq \cdots$ be a descending chain of right annihilators. Since $R$ is orthogonally finite, $I = \bigcap I_i$ contains an idempotent $e$ such that $(1-e)I$ contains no nonzero idempotent. We can easily see that there exists an positive integer $j$ such that the right annihilator $(1-e)I_j$ contains no nonzero idempotent. Then there exists $k \geq j$ such that $(1-e)I_k = (1-e)I_{k+1} = \cdots$. Therefore $I_k = eR \oplus (1-e)I_k = eR \oplus (1-e)I_{k+m} = I_{k+m}$ for all $m \geq 0$.\[\square\]

The following example shows that the conditions in (2) of Theorem 2.1...
are not superfluous.

Example 2.2. Let $F$ be a field and let $A = \prod_{i=1}^{\infty} A_i$, where $A_i = F[x]$ is the polynomial ring over $F$. Then $A$ satisfies ACC (resp. DCC) on right annihilators containing no nonzero idempotents, but $A$ is not orthogonally finite.

Next, let $R$ be the subring of $A$ generated by $\bigoplus_{i=1}^{\infty} S_i$ and $1_A$, where $S_i = xF[x]$ is the ideal of $A_i$ generated by $x$ for all $i = 1, 2, \ldots$. Then $R$ is a ring with only idempotents 0 and $1_A$, but $R$ does not satisfy ACC (resp. DCC) on right annihilators containing no nonzero idempotents.

As an immediate corollary of Theorem 2.1, we have the following.

Corollary 2.3. For a ring $R$ the following statements are equivalent:

1. Every nonzero right annihilators in $R$ contains a nonzero idempotent and $R$ is orthogonally finite;
2. $R$ is an orthogonally finite Baer ring;
3. Every nonzero right annihilator in $R$ contains a nonzero idempotent and $R$ satisfies ACC on right annihilators.

In the proof of [11, Theorem 1], Rangaswamy used the fact that a countable Baer ring has only countably many right annihilators. More generally, if every right annihilator of a countable ring $R$ is a right annihilator of some finite subset of $R$, then $R$ has only countably many right annihilators. In fact a countable ring satisfying the descending chain condition on right annihilators has this property. The following proposition was proved by Faith [3, Corollary 2]. However we give a more direct proof of it.

Proposition 2.4. Then the following statements are equivalent:

1. A ring $R$ satisfies DCC on right annihilators;
2. For each nonempty subset $S$ of $R$, there exists a finite subset $S'$ of $S$ such that $r(S) = r(S')$.

Proof. (1) $\Rightarrow$ (2). Let $S$ be a nonempty subset of $R$. Let $a_1 \in S$. If $r(S) \subseteq r(a_1)$, there exists $a_2 \in S$ such that $r(a_1) \supseteq r(a_1, a_2)$. Continuing this process, we obtain a proper descending chain of right annihilators. Since $R$ satisfies DCC on right annihilators, $r(S) = r(a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n \in S$.

(2) $\Rightarrow$ (1). Let $r(S_1) \supseteq r(S_2) \supseteq \cdots$ be a descending chain of right annihilators. We can easily see that $\bigcap_i r(S_i) = r(\bigcup_i S_i)$. By hypothesis, there exists a finite subset $S'$ of $\bigcup_i S_i$ such that $r(S') = r(\bigcup_i S_i)$. Since
If a countable ring $R$ satisfies ACC or DCC on right annihilators, then $R$ has only countably many right annihilators.

Proof. If $R$ satisfies DCC on right annihilators, then the set of right annihilators is countable by Proposition 2.4. If $R$ satisfies ACC on right annihilators, then $R$ satisfies DCC on left annihilators. Then the set of left annihilators is countable. Since there is a one to one correspondence between the set of left annihilators and the set of right annihilators, the set of right annihilators is also countable.

3. $p$-injective rings

Let $R$ be a ring with identity. A right $R$-module $M$ is said to be $p$-injective if given any principal right ideal $I$ and any $R$-homomorphism $\sigma : I \to M$, there exists an $R$-homomorphism $\hat{\sigma} : R \to M$ that extends $\sigma$. A ring $R$ is called a right $p$-injective ring if $R$ is $p$-injective. This notion was first introduced by Ikeda and Nakayama[7]. It is easily seen that a von Neumann regular ring is nonsingular and right $p$-injective. For other examples of nonsingular $p$-injective rings, see [6]. The Jacobson radical of $R$ is denoted by $J(R)$.

The following generalizes [9, Theorem 8].

**Theorem 3.1.** Let $R$ be a right nonsingular right $p$-injective ring. Then the following are equivalent:

1. $R$ satisfies ACC on right annihilators;
2. $R$ has a finite right uniform dimension;
3. $R$ is a semisimple Artinian ring.

Proof. (1) $\Rightarrow$ (3). By [7], a ring $R$ is right $p$-injective if and only if every principal left ideal of $R$ is a left annihilator. Since $R$ satisfies ACC on right annihilators, $R$ satisfies DCC on left annihilators. Hence $R$ satisfies DCC on principal left ideals, and hence $R$ is a right perfect ring. By [12, Corollary 8.5.4, p.190], $R$ is semiprimary. Suppose that $J(R) \neq 0$. Let $r(a)$ be maximal in $\{r(x) \mid 0 \neq x \in J(R)\}$. Since $R$ is left nonsingular, $r(a)$ is not essential. Hence we can choose a nonzero element $b \in R$ such that $r(a) \cap bR = 0$. Since $R$ is right semi-artinian, we may assume that $bR$ is a minimal right ideal of $R$. Since $ab \neq 0$, $bR \cong abR$. Since $r(ab) \supseteq r(b)$ and $r(b)$ is maximal right ideal of $R$, $r(ab)$ is also maximal right ideal of $R$. Therefore $r(ab)$ is a maximal left ideal of $R$. By [9, Theorem 8], $R$ is semisimple Artinian.
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Let \( r(b) = r(ab) \). Hence \( Rb = l(r(b)) = l(r(ab)) = Rab \). Hence \( b = cab \) for some \( c \in R \), and so \( b \in r(a - aca) \). Since \( r(a) \) is maximal in \( \{ r(x) \mid 0 \neq x \in J(R) \} \), we conclude that \( a - aca = 0 \). Then \( J(R) \) contains a nonzero idempotent \( ac \), a contradiction. Therefore \( J(R) = 0 \), and hence \( R \) is a semisimple Artinian ring.

\((3) \Rightarrow (2)\) is clear.

\((2) \Rightarrow (1)\). Let \( Q \) denote the maximal right ring of quotients of \( R \). It is well-known that \( Q \) is a von Neumann regular ring. Since \( R_R \) is an essential \( R \)-submodule of \( Q_R \) and since \( R_R \) has finite uniform dimension, \( Q_Q \) is also finite uniform dimension. Since \( Q \) is regular, this implies \( Q \) is Artinian semisimple. Since \( Q \) satisfies ACC on right annihilators, the subring \( R \) also satisfies the same condition.

Using Theorem 3.1, we can slightly generalize [11, Theorem 1] as follows.

**Corollary 3.2.** A countable right p-injective Baer ring is a semisimple Artinian ring.

**Proof.** It is well known that a Baer ring is right and left nonsingular. From his proof of [11, Theorem 1] we know that a countable Baer ring is orthogonally finite. Hence by Corollary 2.3, \( R \) satisfies ACC on right annihilators. Hence the assertion follows from Theorem 3.1.

\(\square\)

A ring \( R \) is said to be of bounded index (of nilpotency) if there exists a positive integer \( n \) such that \( a^n = 0 \) for each nilpotent element \( a \) of \( R \). If \( n \) is the least such integer, we say that \( R \) has index \( n \). For an example, it is well-known that any semiprime PI-ring is of bounded index ([6, Theorem 10.8.2]).

In [6, Proposition 1], we proved that if \( R \) is a semiprime p-injective ring of bounded index, then \( R \) is strongly \( \pi \)-regular. The following corollary shows that the structure of a prime right p-injective ring of bounded index is more simple.

**Corollary 3.3.** Let \( R \) be a prime right p-injective ring of bounded index. Then \( R \) is a simple Artinian ring.

**Proof.** By [5, Proposition 4], \( R \) is right nonsingular. Then, by [5, Proposition 5], \( R \) satisfies ACC on right annihilators. Hence the assertion follows from Theorem 3.1.

\(\square\)
4. Semiprime rings

Let \( R \) be a ring. A family \( \{ S_i \subseteq R \mid i \in I \} \) of subsets of \( R \) is said to be independent with respect to right annihilators if, for any distinct subsets \( J, K \) of \( I \), \( r(\bigcup_{j \in J} S_j) \neq r(\bigcup_{k \in K} S_k) \). The following lemma is trivial.

**Lemma 4.1.** If \( R \) has only countably many right annihilators (resp. right annihilators of ideals), then \( R \) has no infinite family of subsets (resp. ideals) of \( R \) which is independent with respect to right annihilators.

A ring \( R \) is called *quasi-Baer* if the right annihilator of every ideal of \( R \) is generated by an idempotent of \( R \).

**Proposition 4.2.** Let \( R \) be a countable semiprime ring. Then \( R \) is a quasi-Baer ring if and only if \( R \) is a finite direct sum of prime rings.

**Proof.** Obviously a finite direct sum of prime rings is quasi-Baer. Conversely suppose that \( R \) is quasi-Baer. Then, since \( R \) is countable, \( R \) has only countably many right annihilators of ideals. Then by Lemma 4.1, the \( R\)–\( R \)-bimodule \( R \) has finite uniform dimension, say \( n \). Hence \( R \) contains a direct sum \( I_1 \oplus \cdots \oplus I_n \) where each \( I_i \) is a nonzero ideal of \( R \). If we set \( R_k = r(\sum_{i \neq k} I_i) \) for each \( k = 1, \cdots, n \), then each \( R_k \) is a prime ring and \( R = R_1 \oplus \cdots \oplus R_n \). \( \square \)

A ring \( R \) is *biregular* if the principal ideal \( (a) \) generated by every \( a \) has the form \( (e) \) where \( e \) is a central idempotent (see [8, p.210]).

**Corollary 4.3.** A countable biregular quasi-Baer ring is a finite direct sum of simple rings.

**Proof.** Since a biregular ring is semiprime, by Proposition 4.2 \( R \) is a finite direct sum of a prime rings. Obviously a prime biregular ring is a simple ring. \( \square \)

**Proposition 4.4.** Let \( R \) be a countable semiprime ring of bounded index. Then the following are equivalent:

1. \( R \) has only countably many right annihilators;
2. There exists a positive integer \( n \) such that every chain of right annihilators in \( R \) has at most \( n \) proper inclusions;
3. \( R \) satisfies ACC on right annihilators.
Proof. (1) ⇒ (2). By Lemma 4.1, there are uniform ideals \( I_1, \ldots, I_m \) of \( R \) such that \( I_1 \oplus \cdots \oplus I_m \) is an essential (right) ideal of \( R \). We shall show that \( r(I_i) \) is a prime ideal of \( R \) for each \( i = 1, 2, \ldots, m \). Let \( I, J \) be ideals of \( R \) such that \( IJ \subseteq r(I_i) \) and suppose that \( I_i, J \neq 0 \). Then \( (I_1 \oplus \cdots \oplus I_{i-1} \oplus I_i \oplus I_{i+1} \oplus \cdots \oplus I_m)I, J = 0 \). Since \( (I_1 \oplus \cdots \oplus I_{i-1} \oplus I_i \oplus I_{i+1} \oplus \cdots \oplus I_m) \) is an essential right ideal of \( R \) and since \( R \) is nonsingular by [5, Proposition 4], we have \( I_i, J = 0 \). Hence each \( r(I_i) \) is a prime ideal of \( R \). We also have that \( \bigcap_{i=1}^n r(I_i) = r(I_1 \oplus \cdots \oplus I_m) = 0 \). Hence \( R \) is embedded in \( \bigoplus_{i=1}^n R/r(I_i) \). Let \( k \) be the nilpotency index of \( R \). Then the index of each \( R/r(I_i) \) is equal to or less than \( k \) by [1, Lemma 3]. Hence, by [5, Proposition 5], every chain of right annihilators in \( R/r(I_i) \) has at most \( k \) proper inclusions. Since \( R \) is embedded in \( \bigoplus_i R/r(I_i) \), every chain of right annihilators in \( R \) has at most \( mk \) proper inclusions.

(2) ⇒ (3). This is trivial.

(3) ⇒ (1). This follows from Corollary 2.5. \( \square \)

Lanski [10, Theorem 2] proved that if \( R \) is a semiprime PI-ring with infinite center \( C \), then \( |R| \leq 2^{|C|} \), where \( |R| \) (resp. \( |C| \)) denotes the cardinality of \( R \) (resp. \( C \)).

The following shows that if \( C \) is countable and if \( C \) has only countably many right annihilators then \( |R| = |C| \).

**Corollary 4.5.** Let \( R \) be a semiprime PI-ring with center \( C \). Then the following are equivalent:

1. \( R \) is countable and it has only countably many right annihilators;
2. \( C \) is countable and it has only countably many right annihilators;
3. \( R \) is a countable semiprime Goldie ring.

**Proof.** (1) ⇒ (2) is trivial.

(2) ⇒ (3). Since \( R \) is semiprime, \( C \) has no nonzero nilpotent elements. By Proposition 4.4, \( C \) satisfies ACC on annihilators in \( C \). Then by Formanek [4, Theorem 9] \( R \) is semiprime Goldie. Now we can easily see that \( R \) is also countable.

(3) ⇒ (1). By [6, Theorem 10.8.2] \( R \) is of bounded index. Hence this follows from Proposition 4.4. \( \square \)

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References


Yasuuki Hirano, Department of Mathematics, Okayama University, Okayama 700-8530, Japan
E-mail: yhirano@math.okyama-u.ac.jp

Hong Kee Kim, Department of Mathematics, Gyeongsang National University, Jinju 660-701, Korea
E-mail: hkkim@gaechuk.gsu.ac.kr