**k-TH ROOTS OF p-HYPONORMAL OPERATORS**

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**Abstract.** In this paper we prove that if \(T\) is a \(k\)-th root of a \(p\)-hyponormal operator when \(T\) is compact or \(T^n\) is normal for some integer \(n > k\), then \(T\) is (generalized) scalar, and that if \(T\) is a \(k\)-th root of a semi-hyponormal operator and have the property \(\sigma(T)\) is contained in an angle \(\angle < \frac{2\pi}{k}\) with vertex in the origin, then \(T\) is subscalar.

**1. Introduction**

Let \(H\) and \(K\) be complex Hilbert spaces and let \(\mathcal{L}(H, K)\) denote the space of all bounded linear operators from \(H\) to \(K\). If \(H = K\), we write \(\mathcal{L}(H)\) in place of \(\mathcal{L}(H, K)\).

A bounded linear operator \(S\) on \(H\) is called *scalar* of order \(m\) if it has a spectral distribution of order \(m\), i.e., if there is a continuous unital morphism of topological algebras

\[
\Phi : C^m_0(\mathbb{C}) \rightarrow \mathcal{L}(H)
\]

such that \(\Phi(z) = S\), where as usual \(z\) stands for the identity function on \(\mathbb{C}\) and \(C^m_0(\mathbb{C})\) stands for the space of compactly supported functions on \(\mathbb{C}\), continuously differentiable of order \(m\), \(0 \leq m \leq \infty\). An operator is *subscalar* if it is similar to the restriction of a scalar operator to a closed invariant subspace.

Let \(d\mu(z)\), or simply \(d\mu\), denote the planar Lebesgue measure. Let \(D\) be a bounded open disc in \(\mathbb{C}\). We shall denote by \(L^2(D, H)\) the Hilbert

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\(^(*)\) This work was supported by Korea Research Foundation Grant(KRF-2001-050-D0001).
space of measurable functions $f : D \to H$, such that

$$\|f\|_{2,D} = \left( \int_D \|f(z)\|^2 \, d\mu(z) \right)^{\frac{1}{2}} < \infty.$$  

The space of functions $f \in L^2(D, H)$ which are analytic functions in $D$ (i.e., $\partial f = 0$) is defined by

$$A^2(D, H) = L^2(D, H) \cap \mathcal{O}(D, H),$$

where $\mathcal{O}(D, H)$ denotes the Fréchet space of $H$-valued analytic functions on $D$ with respect to uniform topology. $A^2(D, H)$ is called the Bergman space for $D$. Let us define a Sobolev type space, denoted $W^2(D, H)$.

$W^2(D, H)$ will be the space of those functions $f \in L^2(D, H)$ whose derivatives $\partial f, \partial^2 f$ in the sense of distributions still belong to $L^2(D, H)$. Endowed with the norm $\|f\|_{W^2} = \sum_{i=0}^2 \|\partial^i f\|_{2,D}^2$, $W^2(D, H)$ becomes a Hilbert space contained continuously in $L^2(D, H)$. Now, for $f \in C_0^2(\mathbb{C})$, let $M_f$ denote the operator on $W^2(D, H)$ given by multiplication by $f$. This has a spectral distribution of order 2, defined by the functional calculus

$$\Phi_M : C_0^2(\mathbb{C}) \longrightarrow \mathcal{L}(W^2(D, H)), \quad \Phi_M(f) = M_f.$$

Therefore $M_f$ is a scalar operator of order 2. Consider a bounded open disk $D$ which contains $\sigma(T)$ and the quotient space

$$(1.1) \quad H(D) = W^2(D, H)/\langle T - z \rangle W^2(D, H)$$

endowed with the Hilbert space norm. We denote the class of a vector $f$ or an operator $A$ on $H(D)$ by $\hat{f}$, respectively $\hat{A}$. Let $M_z$ be the operator of multiplication by $z$ on $W^2(D, H)$. As noted above, $M_z$ is a scalar of order 2 and has a spectral distribution $\Phi$. Let $S \equiv \hat{M}_z$. Since $\langle T - z \rangle W^2(D, H)$ is invariant under every operator $M_f, f \in C_0^2(\mathbb{C})$, we infer that $S$ is a scalar operator of order 2 with spectral distribution $\Phi$.

Consider the natural map $V : H \longrightarrow H(D)$ defined by $Vh = \hat{1} \otimes h$, for $h \in H$, where $1 \otimes h$ denotes the constant function identically equal to $h$. In [11], Putinar showed that if $T \in \mathcal{L}(H)$ is a hyponormal operator then $V$ is one-to-one and has closed range such that $VT = SV$, and so $T$ is subscalar of order 2.

An operator $T \in \mathcal{L}(H)$ is said to be $p$-hyponormal, $0 < p \leq 1$, if $(T^*T)^p \geq (TT^*)^p$ where $T^*$ is the adjoint of $T$. If $p = 1$, $T$ is hyponormal and if $p = \frac{1}{2}$, $T$ is called semi-hyponormal. Semi-hyponormal operators were introduced by Xia (see [12]) and there are many works on general $p$-hyponormal operators ([1], [3], [5], [6], [9]).
Löwner-Heinz’s inequality. Let \( A, B \in \mathcal{L}(H) \) be \( A \geq B \geq 0 \) and \( p \in (0, 1] \). Then
\[
A^p \geq B^p.
\]
This inequality gives the following implications:

\[
\begin{align*}
\text{hyponormality} & \implies \text{\( p \)-hyponormality} \ (\frac{1}{2} < p < 1) \\
& \implies \text{semi-hyponormality} \\
& \implies \text{\( p \)-hyponormality} \ (0 < p < \frac{1}{2}).
\end{align*}
\]

It is well known that all the above implications are strict (see [6] and [12]).

In this paper we prove that if \( T \) is a \( k \)-th root of a \( p \)-hyponormal operator when \( T \) is compact or \( T^n \) is normal for some integer \( n > k \), then \( T \) is (generalized) scalar, and that if \( T \) is a \( k \)-th root of a semi-hyponormal operator and has the property \( \sigma(T) \) is contained in an angle < \( 2\pi/k \) with vertex in the origin, then \( T \) is subscalar. These results extend [8, Theorem 4.3].

2. Results

**Theorem 2.1.** Let \( T \) be a \( k \)-th root of a \( p \)-hyponormal operator. If \( T \) is compact or \( T^n \) is normal for some integer \( n > k \), then \( T \) is a (generalized) scalar operator.

**Proof.** First, we claim that \( T^k \) is normal. If \( T \) is compact, then that is straightforward, since \( T^k \) is compact and a compact \( p \)-hyponormal operator is normal ([5, Theorem 2]). If \( T^n \) is normal for some integer \( n > k \), then there exists an \( n \)-nilpotent operator \( T_0 \) and an operator \( T_1 \) which is quasi-similar to a normal operator \( N \) with \( \sigma(T_1) = \sigma(N) \) such that \( T = T_0 \oplus T_1 \) [7, Theorem 3.1]. Consider \( T^k = T_0^k \oplus T_1^k \). Clearly, \( T_0^k \) is nilpotent. Since the only quasi-nilpotent \( p \)-hyponormal operator is the zero operator, \( T_0^k = 0 \). Let \( X \) be a quasi-affinity such that \( T_1^k X = XN^k \). Applying the Putnam-Fuglede theorem for \( p \)-hyponormal operators ([3, Theorem 7]), it follows that \( T_1^k \) is normal. Hence \( T^k \) is normal. Now it follows from [2] and [7, Remark, p.141] that \( T \) is a (generalized) scalar operator. \( \square \)
Corollary 2.2. Let $T$ be a $k$-th root of a $p$-hyponormal operator. If $T$ is compact or $T^n$ is normal for some integer $n > k$, then $T$ has hyperinvariant subspaces.

Proof. Since $T$ is a (generalized) scalar operator by Theorem 2.1, $T$ is decomposable. Hence $T$ has hyperinvariant subspaces. □

Theorem 2.3. Let $T$ be a $k$-th root of a semi-hyponormal operator and have the property $\sigma(T)$ is contained in an angle $< 2\pi/k$ with vertex in the origin. Then $T$ is subscalar of order 2.

We need the following lemmas to prove Theorem 2.3.

Lemma 2.4. ([11, Proposition 2.1]) For every bounded disk $D$ in $\mathbb{C}$ there is a constant $C_D$, such that for an arbitrary operator $T \in \mathcal{L}(H)$ and $f \in W^2(D, H)$ we have

$$
\|(I - P)f\|_{2,D} \leq C_D \left(\|(T - z)^* \partial f\|_{2,D} + \|(T - z)^* \partial^2 f\|_{2,D}\right),
$$

where $P$ denotes the orthogonal projection of $L^2(D, H)$ onto the Bergman space $A^2(D, H)$.

Lemma 2.5. ([9, Lemma 4]) Let $T$ be a semi-hyponormal. Then for a $z \in \mathbb{C}$ and a sequence $f_n \in L^2(D, H)$

$$
\lim_{n \to \infty} \|(T - z)f_n\|_{2,D} = 0 \implies \lim_{n \to \infty} \|(T - z)^* f_n\|_{2,D} = 0.
$$

Proof of Theorem 2.3. Consider a bounded disk $D$ which contains $\sigma(T)$ and $H(D)$ as in (1.1). Then we define the map $V : H \rightarrow H(D)$ by

$$
Vh = 1 \otimes h \left(\equiv 1 \otimes h + (T - z)W^2(D, H)\right),
$$

where $1 \otimes h$ denotes the constant function sending any $z \in D$ to $h$. As mentioned in section 1, to prove Theorem 2.3 it suffices to show that $V$ is one-to-one and has closed range.

Let $h_n \in H$ and $f_n \in W^2(D, H)$ be sequences such that

$$
\lim_{n \to \infty} \|(T - z)f_n + 1 \otimes h_n\|_{W^2} = 0.
$$

Then equation (2.1) implies

$$
\lim_{n \to \infty} \|(T - z)\partial^i f_n\|_{2,D} = 0 \quad \text{for} \quad i = 1, 2.
$$
From (2.2), we get
\[
\lim_{n \to \infty} \| (T^k - z^k) \bar{\partial}^i f_n \|_{2,D} = 0 \quad \text{for} \quad i = 1, 2.
\]
Since \( T^k \) is semi-hyponormal, by Lemma 2.5 we have
\[
\lim_{n \to \infty} \| (T^{*k} - z^k) \bar{\partial}^i f_n \|_{2,D} = 0. \tag{2.3}
\]
Now we claim that
\[
\lim_{n \to \infty} \| (T - z) \bar{\partial}^i f_n \|_{2,D \setminus \sigma(T)} = 0. \tag{2.4}
\]
Indeed, since \( T - z \) is invertible for \( z \in D \setminus \sigma(T) \), the equation (2.2) implies that
\[
\lim_{n \to \infty} \| \bar{\partial}^i f_n \|_{2,D \setminus \sigma(T)} = 0.
\]
Therefore,
\[
\lim_{n \to \infty} \| (T - z)^* \bar{\partial}^i f_n \|_{2,D \setminus \sigma(T)} = 0.
\]
Also, since \( \sigma(T) \) is contained in an angle \( < \frac{2\pi}{k} \) with vertex in the origin, it is clear from the equation (2.3) that
\[
\lim_{n \to \infty} \| (T - z)^* \bar{\partial}^i f_n \|_{2,D} = 0.
\]
Thus Lemma 2.4 and equation (2.4) imply
\[
\lim_{n \to \infty} \| (I - P) f_n \|_{2,D} = 0,
\]
where \( P \) denotes the orthogonal projection of \( L^2(D, H) \) onto \( A^2(D, H) \). Then by (2.1)
\[
\lim_{n \to \infty} \| (T - z)P f_n + 1 \otimes h_n \|_{2,D} = 0.
\]
Let \( \Gamma \) be a curve in \( D \) surrounding \( \sigma(T) \). Then for \( z \in \Gamma \)
\[
\lim_{n \to \infty} \| P f_n(z) + (T - z)^{-1}(1 \otimes h_n) \| = 0, \quad \text{uniformly}.
\]
Hence
\[
\lim_{n \to \infty} \left\| \frac{1}{2\pi i} \int_\Gamma P f_n(z)dz + h_n \right\| = 0.
\]
But by Cauchy’s theorem,
\[
\int_\Gamma P f_n(z)dz = 0.
\]
Thus \( \lim_{n \to \infty} h_n = 0 \). Hence \( V \) is one-to-one and has closed range. This completes the proof. \( \Box \)
Corollary 2.6. Let $T$ be a $k$-th root of a semi-hyponormal operator and have the property that $\sigma(T)$ is contained in an angle $< 2\pi/k$ with vertex in the origin. If $\sigma(T)$ has interior in the plane, then $T$ has a non-trivial invariant subspace.

Proof. The corollary follows from Theorem 2.3 and [4]. □

We say that an operator $T - z$ on the space $\mathcal{O}(D, H)$ has Bishop’s property $(\beta)$ if $T - z$ is one-to-one and has closed range for every disc $D$. Since every subscalar operator has Bishop’s property $(\beta)$ ([10]), from Theorem 2.3 we have the following.

Corollary 2.7. Let $T$ be as in Corollary 2.6. Then $T$ has Bishop’s property $(\beta)$.

Does Theorem 2.3 hold for $k$-th roots of arbitrary $p$-hyponormal operators? A partial answer is given by the following corollary.

Corollary 2.8. Let $T$ be the $k$-th root of a $p$-hyponormal operator $A$, $0 < p < \frac{1}{2}$, such that $0 \notin \sigma(|A|^\frac{1}{2})$. If $\sigma(T)$ is contained in angle $< 2\pi/k$ with vertex in the origin, $T$ is subscalar of order 2.

Proof. Letting $A$ have the polar decomposition $A = U|A|$, it is seen that the operator $S = |A|^\frac{1}{2}U|A|^\frac{1}{2}$ is a semi-hyponormal operator such that $S = |A|^\frac{1}{2}A|A|^{-\frac{1}{2}}$. Since $S = |A|^\frac{1}{2}T^k|A|^{-\frac{1}{2}} = (|A|^\frac{1}{2}T|A|^{-\frac{1}{2}})^k$, $S$ has a $k$-th root $T_0 = |A|^\frac{1}{2}T|A|^{-\frac{1}{2}}$ with spectrum contained in an angle $< 2\pi/k$ with vertex in the origin. Hence $T_0$, and so also $T$, is subscalar of order 2 by Theorem 2.3. □

References

$k$-th roots of $p$-hyponormal operators


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