EXISTENCE OF SOLUTIONS OF FUZZY DELAY
INTEGRODIFFERENTIAL EQUATIONS WITH NONLOCAL
CONDITION

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Abstract. In this paper we prove the existence of solutions of fuzzy delay integro-differential equations with nonlocal condition. The results are obtained by using the fixed point principles.

1. Introduction

Several authors [3-7,11,12] have studied the fuzzy differential equations by using the $H$-differentiability for the fuzzy valued mappings of a real variable whose values are normal, convex, upper semi continuous and compactly supported fuzzy sets in $R^n$. Seikkala [10] defined the fuzzy derivative which is generalization of the Hukuhara derivative in [8]. For the Cauchy problem $x' = f(t, x)$, $x(t_0) = x_0$, the local existence theorems are proved in [11], and the existence theorems under compactness-type conditions are investigated in [12] when the fuzzy valued mapping $f$ satisfies the generalized Lipschitz condition. Park et al [7] studied the fuzzy differential equation with nonlocal condition. Nieto [6] proved an existence theorem for fuzzy differential equations on the metric space $(E^n, D)$. Balachandran and Prakash [2] proved the existence of solutions of fuzzy delay differential equations with nonlocal condition of the form

$$x' = f(t, x(\sigma_1(t)), x(\sigma_2(t)), \ldots, x(\sigma_n(t))), \quad t \in J = [0, a],$$

$$x(0) - g(t_1, t_2, \ldots, t_p, x(\cdot)) = x_0.$$  

In this paper we study the existence of solutions of fuzzy delay integro-differential equations with nonlocal condition of the form

$$x'(t) = f\left(t, x(\sigma_1(t)), \int_0^t h\left(t, s, x(\sigma_2(s)), \int_0^s k(s, \tau, x(\sigma_3(\tau)))d\tau\right)ds\right),$$

$$x(0) - g(t_1, t_2, \ldots, t_p, x(\cdot)) = x_0,$$

where $f : J \times E^n \times E^n \to E^n$, $h : J \times J \times E^n \times E^n \to E^n$ and $k : J \times J \times E^n \to E^n$ are levelwise continuous functions, $g : J^p \times E^n \to E^n$ satisfies the Lipschitz condition

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and \( \sigma_i : J \to J \), \( i = 1, 2, 3 \) are continuous functions, \( \sigma_i(t) \leq t \) for all \( t \in J \). The existence of solutions for non fuzzy case of the problem (1)-(2) has been discussed in [5]. The symbol \( g(t_1, t_2, \cdots t_p, x(\cdot)) \) is used in the sense that in the place of \( \ell' \), we can substitute only elements of the set \( \{ t_1, t_2, \cdots , t_p \} \). For example, \( g(t_1, t_2, \cdots , t_p, x(\cdot)) \) can be defined by the formula
\[
g(t_1, t_2, \cdots , t_p, x(\cdot)) = c_1 x(t_1) + c_2 x(t_2) + \cdots + c_p x(t_p),
\]
where \( c_i (i = 1, 2, \cdots , p) \) are given constants.

2. Preliminaries

Let \( P_K(R^n) \) denote the family of all nonempty, compact, convex subsets of \( R^n \). Addition and scalar multiplication in \( P_K(R^n) \) are defined as usual. Let \( A \) and \( B \) be two nonempty bounded subsets of \( R^n \). The distance between \( A \) and \( B \) is defined by the Hausdorff metric
\[
d(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} ||a - b||, \sup_{b \in B} \inf_{a \in A} ||a - b|| \right\},
\]
where \( || \cdot || \) denote the usual Euclidean norm in \( R^n \). Then it is clear that \( (P_K(R^n), d) \) becomes a metric space. Let \( I = [t_0, t_0 + a] \subset R \) \( (a > 0) \) be a compact interval and let \( E^n \) be the set of all \( u : R^n \to [0, 1] \) such that \( u \) satisfies the following conditions:

(i) \( u \) is normal, that is, there exists an \( x_0 \in R^n \) such that \( u(x_0) = 1 \),
(ii) \( u \) is fuzzy convex, that is, \( u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\} \), for any \( x, y \in R^n \) and \( 0 \leq \lambda \leq 1 \),
(iii) \( u \) is upper semicontinuous,
(iv) \( [u]^0 = \text{cl}\{x \in R^n : u(x) > 0\} \) is compact.

If \( u \in E^n \), then \( u \) is called a fuzzy number, and \( E^n \) is said to be a fuzzy number space. For \( 0 < \alpha \leq 1 \), denote \( [u]^{\alpha} = \{x \in R^n : u(x) \geq \alpha\} \). Then from (i)-(iv), it follows that the \( \alpha \)-level set \( [u]^{\alpha} \in P_K(R^n) \) for all \( 0 \leq \alpha \leq 1 \).

If \( g : R^n \times R^n \to R^n \) is a function, then using Zadeh’s extension principle we can extend \( g \) to \( E^n \times E^n \to E^n \) by the equation
\[
\tilde{g}(u, v)(z) = \sup_{z = g(x, y)} \min\{u(x), v(y)\}.
\]
It is well known that \( [\tilde{g}(u, v)]^{\alpha} = g([u]^{\alpha}, [v]^{\alpha}) \) for all \( u, v \in E^n \), \( 0 \leq \alpha \leq 1 \) and continuous function \( g \). Further, we have \( [u + v]^{\alpha} = [u]^{\alpha} + [v]^{\alpha} \), \( [ku]^{\alpha} = k[u]^{\alpha} \), where \( k \in R \). Define \( D : E^n \times E^n \to [0, \infty) \) by the relation \( D(u, v) = \sup_{0 \leq \alpha \leq 1} d([u]^{\alpha}, [v]^{\alpha}) \), where \( d \) is the Hausdorff metric defined in \( P_K(R^n) \). Then \( D \) is a metric in \( E^n \).

Further we know that [9]

(i) \( (E^n, D) \) is a complete metric space,
(ii) \( D(u + w, v + w) = D(u, v) \) for all \( u, v, w \in E^n \),
(iii) \( D(\lambda u, \lambda v) = |\lambda| D(u, v) \) for all \( u, v \in E^n \) and \( \lambda \in R \).
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It can be proved that $D(u + v, w + z) \leq D(u, w) + D(v, z)$ for $u, v, w$ and $z \in E^n$.

**Definition 2.1.** [3] A mapping $F : I \rightarrow E^n$ is strongly measurable if for all $\alpha \in [0, 1]$ the set-valued map $F_\alpha : I \rightarrow P_K(R^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable when $P_K(R^n)$ has the topology induced by the Hausdorff metric $d$.

**Definition 2.2.** [3] A mapping $F : I \rightarrow E^n$ is said to be integrably bounded if there is an integrable function $h(t)$ such that $\|x(t)\| \leq h(t)$ for every $x(t) \in F_0(t)$.

**Definition 2.3.** The integral of a fuzzy mapping $F : I \rightarrow E^n$ is defined levelwise by $\int_I F(t)dt = \int_I F_\alpha(t)dt$. The set of all $\int_I f(t)dt$ such that $f : I \rightarrow R^n$ is a measurable selection for $F_\alpha$ for all $\alpha \in [0, 1]$.

**Definition 2.4.** [1] A strongly measurable and integrably bounded mapping $F : I \rightarrow E^n$ is said to be integrable over $I$ if $\int_I F(t)dt \in E^n$.

Note that if $F : I \rightarrow E^n$ is strongly measurable and integrably bounded, then $F$ is integrable. Further if $F : I \rightarrow E^n$ is continuous, then it is integrable.

**Proposition 2.1.** Let $F, G : I \rightarrow E^n$ be integrable and $c \in I, \lambda \in R$. Then

(i) $\int_0^{t_0+a} F(t)dt = \int_0^c F(t)dt + \int_{t_0+a}^t F(t)dt$,

(ii) $\int_{I} (F(t) + G(t))dt = \int_{I} F(t)dt + \int_{I} G(t)dt$,

(iii) $\int_{I} \lambda F(t)dt = \lambda \int_{I} F(t)dt$,

(iv) $D(F, G)$ is integrable,

(v) $D \left( \int_{I} F(t)dt, \int_{I} G(t)dt \right) \leq \int_{I} D(F(t), G(t))dt$.

**Definition 2.5** A mapping $F : I \rightarrow E^n$ is Hukuhara differentiable at $t_0 \in I$ if for some $h_0 > 0$ the Hukuhara differences

$$F(t_0 + \Delta t) - h F(t_0), \quad F(t_0) - h F(t_0 - \Delta t)$$

exist in $E^n$ for all $0 < \Delta t < h_0$ and there exists an $F'(t_0) \in E^n$ such that

$$\lim_{\Delta t \to 0^+} D((F(t_0 + \Delta t) - h F(t_0))/\Delta t, F'(t_0)) = 0$$

and

$$\lim_{\Delta t \to 0^+} D((F(t_0) - h F(t_0 - \Delta t))/\Delta t, F'(t_0)) = 0.$$ 

Here $F'(t)$ is called the Hukuhara derivative of $F$ at $t_0$.

**Definition 2.6.** A mapping $F : I \rightarrow E^n$ is called differentiable at a $t_0 \in I$ if, for any $\alpha \in [0, 1]$, the set-valued mapping $F_\alpha(t) = [F(t)]^\alpha$ is Hukuhara differentiable at point $t_0$ with $DF_\alpha(t_0)$ and the family $\{DF_\alpha(t_0) : \alpha \in [0, 1]\}$ define a fuzzy number $F(t_0) \in E^n$. 

If \( F : I \rightarrow E^n \) is differentiable at \( t_0 \in I \), then we say that \( F'(t_0) \) is the fuzzy derivative of \( F(t) \) at the point \( t_0 \).

**Theorem 2.1.** Let \( F : I \rightarrow E^n \) be differentiable. Denote \( F_\alpha(t) = [f_\alpha(t), g_\alpha(t)] \). Then \( f_\alpha \) and \( g_\alpha \) are differentiable and \( [F'(t)]^\alpha = [f_\alpha'(t), g_\alpha'(t)] \).

**Theorem 2.2.** Let \( F : I \rightarrow E^n \) be differentiable and assume that the derivative \( F' \) is integrable over \( I \). Then, for each \( s \in I \), we have

\[
F(s) = F(a) + \int_a^s F'(t)dt.
\]

**Definition 2.7.** A mapping \( f : I \times E^n \rightarrow E^n \) is called levelwise continuous at a point \( (t_0, x_0) \in I \times E^n \) provided, for any fixed \( \alpha \in [0, 1] \) and arbitrary \( \epsilon > 0 \), there exists a \( \delta(\epsilon, \alpha) > 0 \) such that

\[
d([f(t, x)]^\alpha, [f(t_0, x_0)]^\alpha) < \epsilon
\]

whenever \( |t - t_0| < \delta(\epsilon, \alpha) \) and \( d([x]^\alpha, [x_0]^\alpha) < \delta(\epsilon, \alpha) \) for all \( t \in I, x \in E^n \).

**Corollary 2.1** [2] Suppose that \( F : I \rightarrow E^n \) is continuous. Then the function

\[
G(t) = \int_a^t F(s)ds, \quad t \in I
\]

is differentiable and \( G'(t) = F(t) \).

Now, if \( F \) is continuously differentiable on \( I \), then we have the following mean value theorem

\[
D(F(b), F(a)) \leq (b - a) \cdot \sup\{D(F'(t), 0), t \in I\}.
\]

As a consequence, we have that

\[
D(G(b), G(a)) \leq (b - a) \cdot \sup\{D(F(t), 0), t \in I\}.
\]

**Theorem 2.3.** Let \( X \) be a compact metric space and \( Y \) any metric space. A subset \( \Omega \) of the space \( C(X, Y) \) of continuous mappings of \( X \) into \( Y \) is totally bounded in the metric of uniform convergence if and only if \( \Omega \) is equicontinuous on \( X \), and \( \Omega(x) = \{ \phi(x) : \phi \in \Omega \} \) is a totally bounded subset of \( Y \) for each \( x \in X \).

### 3. Main Results

**Definition 3.1.** A mapping \( x : J \rightarrow E^n \) is a solution to the problem (1)-(2) if and only if it is levelwise continuous and satisfies the integral equation

\[
x(t) = x_0 + g(t_1, t_2, \cdots, t_p, x(\cdot)) + \int_0^t f\left(s, x(\sigma_1(s)), \int_0^s h\left(s, \tau, x(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, x(\sigma_3(\theta)))d\theta\right)d\tau\right)ds
\]

for all \( t \in J \).
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Let \( M + Na = b \), a positive number, where

\[
M = \max D \left( f \left( t, x(\sigma_1(t)), \int_0^t h \left( t, s, x(\sigma_2(s)), \int_0^s k(s, \tau, x(\sigma_3(\tau)))d\tau \right) ds \right), 0 \right) \quad \text{and}
\]
\[
N = D(g(t_1, t_2, \cdots, t_p, x(\cdot)), 0), \quad 0 \in E^n.
\]

Let \( Y = \{ \xi \in E^n : H(\xi, x_0) \leq b \} \) be the space of continuous functions with \( H(\xi, \psi) = \sup_{0 \leq t \leq a} D(\xi(t), \psi(t)) \).

**Theorem 3.1.** Assume that:

1. The mapping \( f : J \times Y \to E^n \) is levelwise continuous in \( t \) on \( J \) and there exists a constant \( G_0 \) such that
   \[
   D(f(t, x_1, x_2), f(t, y_1, y_2)) \leq G_0[D(x_1, y_1) + D(x_2, y_2)]
   \]
2. The mapping \( h : J \times J \times Y \to E^n \) is levelwise continuous and there exists a constant \( G_1 \) such that
   \[
   D(h(t, s, x_1, x_2), h(t, s, y_1, y_2)) \leq G_1[D(x_1, y_1) + D(x_2, y_2)]
   \]
3. The mapping \( k : J \times J \times Y \to E^n \) is levelwise continuous and there exists a constant \( G_2 \) such that
   \[
   D(k(t, s, x_1, x_2), k(t, s, y_1, y_2)) \leq G_2 D(x, y)
   \]
4. There exists a constant \( G_3 \) such that for all \( x, y \in Y \) and \( \sigma_i : J \to J, \quad i = 1, 2, 3 \)
   \[
   D(x(\sigma_i(t)), y(\sigma_i(t))) \leq G_3 D(x(t), y(t))
   \]
5. \( g : J^p \times Y \to E^n \) is a function and there exists a constant \( G_4 > 0 \) such that
   \[
   D(g(t_1, t_2, \cdots, t_p, x(\cdot)), g(t_1, t_2, \cdots, t_p, y(\cdot))) \leq G_4 D(x, y).
   \]

Then there exists a unique solution \( x(t) \) of (1)-(2) defined on the interval \( [0, a] \).

**Proof.** Define an operator \( \Phi : Y \to Y \) by

\[
\Phi x(t) = x_0 + g(t_1, t_2, \cdots, t_p, x(\cdot))
\]

\[
+ \int_0^t f \left( s, x(\sigma_1(s)), \int_0^s h \left( s, \tau, x(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, x(\sigma_3(\theta)))d\theta \right) d\tau \right) ds.
\]

First, we show that \( \Phi : Y \to Y \) is continuous whenever \( \xi \in Y \) and that \( H(\Phi \xi, x_0) \leq b. \)

\[
D(\Phi \xi(t + h), \Phi \xi(t))
\]
\[
= D \left( x_0 + g(t_1, t_2, \cdots, t_p, \xi(\cdot)), x_0 + g(t_1, t_2, \cdots, t_p, \xi(\cdot)) \right)
\]
\[
+ \int_0^{t+h} f \left( s, \xi(\sigma_1(s)), \int_0^s h \left( s, \tau, \xi(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, \xi(\sigma_3(\theta)))d\theta \right) d\tau \right) ds,
\]
\[
x_0 + g(t_1, t_2, \cdots, t_p, \xi(\cdot))
\]

\[ \begin{align*} 
&+ \int_0^t f \left( s, \xi(\sigma_1(s)), \int_0^s h \left( s, \tau, \xi(\sigma_2(\tau)) \right), \int_0^\tau k(\tau, \theta, \xi(\sigma_3(\theta))) d\theta \right) d\tau \right) \, ds \\
&\leq D \left( \int_0^{t+h} f \left( s, \xi(\sigma_1(s)), \int_0^s h \left( s, \tau, \xi(\sigma_2(\tau)) \right), \int_0^\tau k(\tau, \theta, \xi(\sigma_3(\theta))) d\theta \right) d\tau \right) \, ds, \\
&\quad + \int_0^t \left( \int_0^s f \left( s, \xi(\sigma_1(s)), \int_0^s h \left( s, \tau, \xi(\sigma_2(\tau)) \right), \int_0^\tau k(\tau, \theta, \xi(\sigma_3(\theta))) d\theta \right) d\tau \right) \, ds, \quad x_0 \\
&\leq \int_0^{t+h} D \left( f \left( s, \xi(\sigma_1(s)), \int_0^s h \left( s, \tau, \xi(\sigma_2(\tau)) \right), \int_0^\tau k(\tau, \theta, \xi(\sigma_3(\theta))) d\theta \right) d\tau \right) \, ds, \quad 0 \\
&\leq hM \to 0 \text{ as } h \to 0. 
\end{align*} \]

That is, the map \( \Phi \) is continuous. Now

\[
D(\Phi \xi(t), x_0) = D(x_0 + g(t_1, t_2, \ldots, t_p, \xi(\cdot))) + \int_0^t f \left( s, \xi(\sigma_1(s)), \int_0^s h \left( s, \tau, \xi(\sigma_2(\tau)) \right), \int_0^\tau k(\tau, \theta, \xi(\sigma_3(\theta))) d\theta \right) d\tau \, ds \\
\leq D \left( f \left( s, \xi(\sigma_1(s)), \int_0^s h \left( s, \tau, \xi(\sigma_2(\tau)) \right), \int_0^\tau k(\tau, \theta, \xi(\sigma_3(\theta))) d\theta \right) d\tau \right) \, ds, \quad x_0 \\
\leq hM \to 0 \text{ as } h \to 0.
\]

Thus \( \Phi \) is a mapping from \( Y \) into \( Y \). Since \( C([0, a], E^n) \) is a complete metric space with the metric \( H \), we only show that \( Y \) is a closed subset of \( C([0, a], E^n) \). Let \( \{ \psi_n \} \) be a sequence in \( Y \) such that \( \psi_n \to \psi \in C([0, a], E^n) \) as \( n \to \infty \). Then

\[
D(\psi(t), x_0) \leq D(\psi(t), \psi_n(t)) + D(\psi_n(t), x_0),
\]

that is,

\[
H(\psi, x_0) = \sup_{0 \leq t \leq a} D(\psi(t), x_0) \leq H(\psi, \psi_n) + H(\psi_n, x_0) \leq \epsilon + b
\]

for sufficiently large \( n \) and arbitrary \( \epsilon > 0 \). So \( \psi \in Y \). This implies that \( Y \) is closed subset of \( C([0, a], E^n) \). Therefore \( Y \) is a complete metric space.

By using Proposition 2.1 and assumptions (i)-(v), we will show that \( \Phi \) is a contraction mapping. For \( \xi, \psi \in Y \),

\[
D(\Phi \xi(t), \Phi \psi(t)) = D \left( x_0 + g(t_1, t_2, \ldots, t_p, \xi(\cdot)) \right)
\]
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\[ + \int_0^t f \left( s, \xi(\sigma_1(s)), \int_0^s h \left( s, \tau, \xi(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, \xi(\sigma_3(\theta)))d\theta \right) d\tau \right) ds, \]

\[ x_0 + g(t_1, t_2, \ldots, t_p, \psi(\cdot)) \]

\[ + \int_0^t f \left( s, \psi(\sigma_1(s)), \int_0^s h \left( s, \tau, \psi(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, \psi(\sigma_3(\theta)))d\theta \right) d\tau \right) ds \]

\[ \leq D \left( g(t_1, t_2, \ldots, t_p, \xi(\cdot)), g(t_1, t_2, \ldots, t_p, \psi(\cdot)) \right) \]

\[ + \int_0^t D \left( \int_0^s h \left( s, \tau, \xi(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, \xi(\sigma_3(\theta)))d\theta \right) d\tau \right) ds, \]

\[ f \left( s, \psi(\sigma_1(s)), \int_0^s h \left( s, \tau, \psi(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, \psi(\sigma_3(\theta)))d\theta \right) d\tau \right) ds \]

\[ \leq G_4 D(\xi(\cdot), \psi(\cdot)) + G_0 \int_0^t D(\xi(\sigma_1(s)), \psi(\sigma_1(s)))ds \]

\[ + G_0 \int_0^t \left( \int_0^s h \left( s, \tau, \xi(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, \xi(\sigma_3(\theta)))d\theta \right) d\tau \right) ds, \]

\[ \int_0^s h \left( s, \tau, \psi(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, \psi(\sigma_3(\theta)))d\theta \right) d\tau \right) ds \]

\[ \leq G_4 D(\xi(\cdot), \psi(\cdot)) + G_0 G_3 \int_0^t D(\xi(s), \psi(s))ds + G_0 G_1 G_3 \int_0^t \int_0^s D(\xi(\tau), \psi(\tau))d\tau ds \]

\[ + G_0 G_1 G_2 G_3 \int_0^t \int_0^s \int_0^\tau D(\xi(\theta), \psi(\theta))d\theta d\tau ds. \]

Then we obtain

\[ H(\Phi_x, \Phi_y) \leq \sup_{0 \leq t \leq a} \left\{ G_4 D(\xi(\cdot), \psi(\cdot)) + G_0 G_3 \int_0^t D(\xi(s), \psi(s))ds \right. \]

\[ + G_0 G_1 G_3 \int_0^t \int_0^s D(\xi(\tau), \psi(\tau))d\tau ds \]

\[ + G_0 G_1 G_2 G_3 \int_0^t \int_0^s \int_0^\tau D(\xi(\theta), \psi(\theta))d\theta d\tau ds \right\} \]

\[ \leq G_4 D(\xi(\cdot), \psi(\cdot)) + aG_0 G_3 D(\xi(t), \psi(t)) \]

\[ + a^2 G_0 G_1 G_3 D(\xi(t), \psi(t)) + a^3 G_0 G_1 G_2 G_3 D(\xi(t), \psi(t)) \]

\[ \leq pH(\xi, \psi), \]

where the constant \( p = G_4 + G_0 G_3 a + G_0 G_1 G_3 a^2 + G_0 G_1 G_2 G_3 a^3 \). Taking sufficiently small \( a \) such that \( p < 1 \), we obtain \( \Phi \) to be a contraction mapping. Therefore \( \Phi \) has a unique fixed point \( x \in C([0, a], E^n) \) such that \( \Phi x = x \), that is,

\[ x(t) = x_0 + g(t_1, t_2, \ldots, t_p, \psi(\cdot)) \]
Theorem 3.2. Let $f, h, k, \sigma$ and $g$ be as in Theorem 3.1. Denote by $x(t, x_0), y(t, y_0)$ the solutions of equation (1) corresponding to $x_0, y_0$, respectively. Then there exists constant $q > 0$ such that

$$H(x(\cdot, x_0), y(\cdot, y_0)) \leq qD(x_0, y_0)$$

for any $x_0, y_0 \in E^n$ and $q = 1/(1-p)$.

Proof. Let $x(t, x_0), y(t, y_0)$ be solutions of equations (1) corresponding to $x_0, y_0$, respectively. Then

$$D(x(t, x_0), y(t, y_0))$$

$$= D(x_0 + g(t_1, t_2, \ldots, t_p, x(\cdot))$$

$$+ \int_0^t f \left( s, x(\sigma_1(s)), \int_0^s h \left( s, \tau, x(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, x(\sigma_3(\theta))) d\theta d\tau \right) ds, y_0 + g(t_1, t_2, \ldots, t_p, y(\cdot))$$

$$+ \int_0^t f \left( s, y(\sigma_1(s)), \int_0^s h \left( s, \tau, y(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, y(\sigma_3(\theta))) d\theta d\tau \right) ds \right)$$

$$\leq D(x_0, y_0) + D(g(t_1, t_2, \ldots, t_p, x(\cdot)), g(t_1, t_2, \ldots, t_p, y(\cdot)))$$

$$+ \int_0^t D f \left( s, x(\sigma_1(s)), \int_0^s h \left( s, \tau, x(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, x(\sigma_3(\theta))) d\theta d\tau \right) ds, y_0 + g(t_1, t_2, \ldots, t_p, y(\cdot))$$

$$+ \int_0^t f \left( s, y(\sigma_1(s)), \int_0^s h \left( s, \tau, y(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, y(\sigma_3(\theta))) d\theta d\tau \right) ds \right)$$

$$\leq D(x_0, y_0) + G_4D(x(\cdot), y(\cdot)) + G_0G_3 \int_0^t D(x(s), y(s)) ds$$

$$+ G_0G_1G_3 \int_0^t \int_0^s D(x(\tau), y(\tau)) d\tau ds + G_0G_1G_2G_3 \int_0^t \int_0^s \int_0^\tau D(x(\theta), y(\theta)) d\theta d\tau d\sigma ds.$$
Proof. Since \( f, h, k \) are continuous and bounded and \( g \) is a continuous function there exists \( r \geq 0 \) such that
\[
D \left( f \left( t, x(\sigma_1(t)), \int_0^t h \left( s, x(\sigma_2(s)), \int_0^s k(s, \tau, x(\sigma_3(\tau))) d\tau \right) ds \right), \hat{0} \right) \leq r, \quad t \in J, x \in E^n.
\]

Let \( B \) be a bounded set in \( C(J, E^n) \). The set \( \Phi B = \{ \Phi x : x \in B \} \) is totally bounded if and only if it is equicontinuous and for every \( t \in J \), the set \( \Phi B(t) = \{ \Phi x(t) : t \in J \} \) is a totally bounded subset of \( E^n \). For \( t_0, t_1 \in J \) with \( t_0 \leq t_1 \) and \( x \in B \) we have that
\[
D(\Phi x(t_0), \Phi x(t_1)) = D \left( x_0 + g(t_1, t_2, \ldots, t_p, x(\cdot)) \right)
\]
\[
+ \int_0^{t_0} f \left( s, x(\sigma_1(s)), \int_0^s h \left( s, \tau, x(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, x(\sigma_3(\theta))) d\theta \right) d\tau \right) ds,
\]
\[
x_0 + g(t_1, t_2, \ldots, t_p, x(\cdot))
\]
\[
+ \int_{t_0}^{t_1} f \left( s, x(\sigma_1(s)), \int_0^s h \left( s, \tau, x(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, x(\sigma_3(\theta))) d\theta \right) d\tau \right) ds
\]
\[
\leq D \left( \int_0^{t_0} f \left( s, x(\sigma_1(s)), \int_0^s h \left( s, \tau, x(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, x(\sigma_3(\theta))) d\theta \right) d\tau \right) ds,
\]
\[
\int_0^{t_1} f \left( s, x(\sigma_1(s)), \int_0^s h \left( s, \tau, x(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, x(\sigma_3(\theta))) d\theta \right) d\tau \right) ds
\]
\[
\leq D \left( f \left( s, x(\sigma_1(s)), \int_0^s h \left( s, \tau, x(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, x(\sigma_3(\theta))) d\theta \right) d\tau \right), \hat{0} \right) ds
\]
\[
\leq |t_1 - t_0| \sup \left\{ D \left( \int_0^1 f \left( t, x(\sigma_1(t)), \int_0^t h \left( t, s, x(\sigma_2(s)), \int_0^s k(s, \tau, x(\sigma_3(\tau))) d\tau \right) ds \right), \hat{0} \right) \right\}
\]
\[
\leq |t_1 - t_0| \cdot r.
\]

This shows that \( \Phi B \) is equicontinuous. Now, for \( t \in J \) fixed, we have
\[
D(\Phi x(t), \Phi x(t')) \leq |t - t'| \cdot r, \quad \text{for every } t' \in J, x \in B.
\]

Consequently, the set \( \{ \Phi x(t) : x \in B \} \) is totally bounded in \( E^n \). By Ascoli’s theorem we conclude that \( \Phi B \) is a relatively compact subset of \( C(J, E^n) \). Then \( \Phi \) is compact, that is, \( \Phi \) transforms bounded sets into relatively compact sets.

We know that \( x \in C(J, E^n) \) is a solution of (1)-(2) if and only if \( x \) is a fixed point of the operator \( \Phi \) defined by (4).

Now, in the metric space \( (C(J, E^n), H) \), consider the ball
\[
B = \{ \xi \in C(J, E^n), H(\xi, \hat{0}) \leq m \}, \quad m = a \cdot r.
\]

Thus, \( \Phi B \subset B \). Indeed, for \( x \in C(J, E^n) \),
\[
D(\Phi x(t), \Phi x(0)) = D \left( x_0 + g(t_1, t_2, \ldots, t_p, x(\cdot)) \right)
\]
\[ + \int_0^t f \left( s, x(\sigma_1(s)), \int_0^s h \left( s, \tau, x(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, x(\sigma_3(\theta))) d\theta \right) d\tau \right) ds, \\
\quad x_0 + g(t_1, t_2, \cdots, t_p, x(\cdot)) \]
\[ \leq \int_0^t D \left( f \left( s, x(\sigma_1(s)), \int_0^s h \left( s, \tau, x(\sigma_2(\tau)), \int_0^\tau k(\tau, \theta, x(\sigma_3(\theta))) d\theta \right) d\tau \right), \hat{0} \right) ds \]
\[ \leq |t| \cdot r \leq a \cdot r. \]

Therefore, defining \( \hat{0} : J \to E^n \), \( \hat{0}(t) = \hat{0}, \ t \in J \) we have
\[ H(\Phi x, \Phi \hat{0}) = \sup \{ D(\Phi x(t), \Phi \hat{0}(t)) : t \in J \}. \]

Therefore \( \Phi \) is compact and, in consequence, it has a fixed point \( x \in B \). This fixed point is a solution of the initial value problem (1)-(2). \( \square \)

**References**


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