AVERAGE SHADOWING PROPERTIES
ON COMPACT METRIC SPACES

JONG-JIN PARK AND YONG ZHANG

Abstract. We prove that if a continuous surjective map $f$ on a compact metric space $X$ has the average shadowing property, then every point $x$ is chain recurrent. We also show that if a homeomorphism $f$ has more than two fixed points on $S^1$, then $f$ does not satisfy the average shadowing property. Moreover, we construct a homeomorphism on a circle which satisfies the shadowing property but not the average shadowing property. This shows that the converse of the theorem 1.1 in [6] is not true.

1. Introduction

The shadowing property (also called the pseudo-orbit tracing property) is one of the most important notions in dynamical systems (see [1]). In [2], Blank introduced the notion of average-shadowing property (see [3]). In [5], Sakai proved that, on a closed $C^\infty$ surface, the $C^1$ interior of the set of $C^1$ diffeomorphisms with the average-shadowing property is characterized by the set of Anosov diffeomorphisms. In [6], Zhang proved that whenever a homeomorphism $f$ on a compact metric space $X$ has the average-shadowing property, every point $x$ in $X$ is chain recurrent.

In this paper, we extend the property of homeomorphisms of the theorem 1.1 in [6] to the property of continuous surjective maps.

Theorem. [6] If a homeomorphism $f$ on a compact metric space $X$ has the average shadowing property, then every point $x$ is chain recurrent. Moreover $f$ has only one chain component which is the whole space.

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And we show that if a homeomorphism $f$ has more than two fixed points on $S^1$, then $f$ does not satisfy the average shadowing property:

**Theorem.** Let $S^1$ be a circle and let $f : S^1 \to S^1$ be a self-homeomorphism. If $N[\text{fix}(f)] \geq 2$, then $f$ does not satisfy the average shadowing property on $S^1$.

Moreover, we show that the converse of the theorem 1.1 in [6] is not true by constructing a homeomorphism on a circle which satisfy the shadowing property, but not the average shadowing property.

2. Notions

Let $(X, d)$ be a compact metric space and let $f : X \to X$ be a homeomorphism of $X$ onto itself. A subset $\text{Fix}(f)$ of $X$ is called a fixed point set of $f$ on $X$. Let $N[\text{Fix}(f)]$ denote the number of the subset $\text{Fix}(f)$. A sequence $\{x_n\}_{n \in \mathbb{Z}}$ is called an orbit of $f$ if $x_{n+1} = f(x_n)$ for all $n \in \mathbb{Z}$ and a $\delta$-pseudo-orbit of $f$ if

$$d(f(x_n), x_{n+1}) \leq \delta,$$

for all $n \in \mathbb{Z}$.

We say that the homeomorphism $f$ has the shadowing property if for each $\epsilon > 0$ there exists $\delta$ such that every $\delta$-pseudo-orbit $\{x_n\}_{n \in \mathbb{Z}}$ is $\epsilon$-shadowed by an orbit $\{f^n(y)\}_{n \in \mathbb{Z}}$ of for some $y \in X$, i.e.

$$d(f^n(y), x_n) \leq \epsilon$$

for all $n \in \mathbb{Z}$.

Let $x, y \in X$ be given. We say $x \xrightarrow{f} y$ if and only if for each $\delta > 0$, there is a $\delta$-pseudo-orbit $\{x_l\}_{l=0}^l$ of some length $l+1$ of $f$, such that $x = x_0, x_1, \ldots, x_l = y$. It is said that $x \xrightarrow{\hat{L}} y$ if and only if $x \xrightarrow{L} y$ and $y \xrightarrow{L} x$. It is denoted $R(f) = \{x \in X | x \xrightarrow{\hat{L}} x\}$. It is easy to see that $\hat{L}$ is an equivalent relation on $R(f)$. It is called an equivalent class with respect to $L$ a chain component of $f$.

For $\delta > 0$, a sequence $\{x_i\}_{i=-\infty}^\infty$ of points in $X$ is called an $\delta$-average-pseudo-orbit of $f$ if there is a natural number $N = N(\delta) > 0$ such that for all $n \geq N$, and $k \in \mathbb{Z}$,

$$\frac{1}{n} \sum_{i=1}^{n} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$

The average-pseudo-orbits are a certain generalization of the notion of pseudo-orbits. It is said that $f$ has the average-shadowing property if
for all $\epsilon > 0$, there exists $\delta > 0$ such that every $\delta$-average-pseudo-orbit $\{x_i\}_{i=-\infty}^{\infty}$ is $\epsilon$-shadowed in average by some $z \in X$, that is
\[
\lim_{n \to \infty} \sup_{z \in X} \frac{1}{n} \sum_{i=1}^{n} \left| d(f^i(z), x_i) \right| < \epsilon.
\]

Let $(X, d)$ be a compact metric space and let $f$ be a continuous map of $X$ onto itself. For $\delta > 0$, a sequence $\{x_i\}_{i=0}^{\infty}$ of points in $X$ is called a $\delta$-average pseudo-orbit of $f$ if there is a natural number $N = N(\delta) > 0$ such that for all $n \geq N$ and $k \geq 0$,
\[
\frac{1}{n} \sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta.
\]
We say that $f$ has the average shadowing property if there is a metric $d$ for $X$ with the following property: for every $\epsilon > 0$, there is $\delta > 0$ such that every $\delta$-average-pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ is $\epsilon$-shadowed in average by some point $y \in X$; that is
\[
\lim_{n \to \infty} \sup_{y \in X} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(y), x_i) < \epsilon.
\]
We use by $B(x, \epsilon)$ the open ball with the center $x$ and the radius $\epsilon$.

3. Average shadowing property

Consider a circle $S^1$ with coordinate $x \in [0, 1)$, and we denote by $d$ the metric on $S^1$ induced by the usual distance on the real line. When we study the theory of shadowing and average shadowing, usually we only consider the homeomorphisms on $S^1$ which preserve orientation.

Let $\Pi(x) : \mathbb{R} \to S^1$ be the covering projection defined by the relations
\[
\Pi(x) \in [0, 1) \text{ and } \Pi(x) \equiv x(x \text{ mod } 1)
\]
with respect to the considered coordinates on $S^1$.

Let $f : S^1 \to S^1$ be a homeomorphism and let a homeomorphism $F : \mathbb{R} \to \mathbb{R}$ be a lifting of $f$.

**Theorem 3.1.** Let $S^1$ be a circle and let $f : S^1 \to S^1$ be a self-homeomorphism. If $N[fix(f)] \geq 2$, then $f$ does not satisfy the average shadowing property on $S^1$.

**Proof.** Let $S^1$ be a circle and let $f : S^1 \to S^1$ be a homeomorphism. Take two fixed points $\{a, b\} \in Fix(f)$ and $\epsilon > 0$ such that $\min\{d(a, b), d(b, a)\} > 3\epsilon$. We denote $D$ by the diameter of $S^1$, that
is, $D = \max_{(x, y) \in S^1 \times S^1} d(x, y)$. Consider $\delta > 0$. Take a natural number $N$ such that $\frac{3D}{N} < \delta$. Define a sequence $\{x_i\}_{i=-\infty}^{\infty}$ by

$$x_i = \begin{cases} a, & \text{if } 0 \leq i \leq N \\ a, & \text{if } 3 \cdot 2^j \cdot N + 1 \leq i \leq 3 \cdot 2^{j+1} \cdot N, \ j = 0, 2, 4, \ldots \\ b, & \text{if } N + 1 \leq i \leq 3N \\ b, & \text{if } 3 \cdot 2^j \cdot N + 1 \leq i \leq 3 \cdot 2^{j+1} \cdot N, \ j = 1, 3, 5, \ldots \end{cases}$$

and

$$x_i = \begin{cases} a, & \text{if } -3N \leq i \leq -N - 1 \\ a, & \text{if } -3N \cdot 2^j + 1 \leq i \leq -3N \cdot 2^{j+1} - 1, \ j = 2, 4, 6, \ldots \\ b, & \text{if } -N \leq i \leq -1 \\ b, & \text{if } -3N \cdot 2^j + 1 \leq i \leq -3N \cdot 2^j - 1, \ j = 0, 1, 3, 5, \ldots \end{cases}$$

Then it is easy to see that for $n > N$ and $k \in \mathbb{Z}$,

$$\frac{1}{n} \sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \frac{1}{n} \cdot \frac{n}{N} \cdot 3D < \delta.$$

Thus $\{x_i\}_{i=0}^{\infty}$ is a $\delta$-average-pseudo orbit of $f$. We assume that there is a point $z$ in $S^1$ such that $\{x_i\}_{i=-\infty}^{\infty}$ is $\epsilon$-shadowed in average by $z$. Then there is a natural number $t$ and a fixed point $c$ of $f$ such that for $n > t$, $f^n(z) \in B(c, \epsilon)$ and since $d(a, b) > 3\epsilon$,

$$d(f^n(z), a) > \epsilon \text{ or } d(f^n(z), b) > \epsilon.$$

Hence

$$\lim_{n \to \infty} \sup_{i=1}^{n} \frac{1}{n} \sum_{i=1}^{n} d(f^i(z), x_i) \geq \epsilon.$$

It is a contradiction and so we complete the proof of Theorem 3.1.

The following Corollary 3.2 shows that the converse of the theorem 1.1 in [6] is not true.

**Corollary 3.2.** There is a homeomorphism $f$ on $S^1$ satisfying following:

1. any point $x$ in $S^1$ is chain recurrent;
2. $f$ does not satisfy the average shadowing property.

**Proof.** Let $F : [0, 1] \to [0, 1]$ be a homeomorphism defined by

$$F(t) = \begin{cases} t + \left(\frac{1}{2} - t\right)t & \text{if } 0 \leq t \leq \frac{1}{2} \\ t + (1-t)(t - \frac{1}{2}) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$
\[ F \text{ induces a homeomorphism } f : S^1 \to S^1. \text{ Obviously } a = \Pi(0) \text{ and } b = \Pi(1/2) \text{ are fixed points of } f. \text{ Then any point } x \text{ in } S^1 \text{ is chain recurrent of } f \text{ and by Theorem 3.1, } f \text{ does not satisfy the average shadowing property.} \]

The following Remark shows that there is a homeomorphism on \( S^1 \) which has the shadowing property, but not the average shadowing property.

**Remark 3.3.** Let \( F : [0, 1] \to [0, 1] \) be a homeomorphism defined by
\[
F(t) = \begin{cases} 
   t + (1/2 - t)t & \text{if } 0 \leq t \leq 1/2 \\
   t - (t - 1/2)(1 - t) & \text{if } 1/2 \leq t \leq 1.
\end{cases}
\]

\( F \) induces a homeomorphism \( f : S^1 \to S^1. \text{ Then } a = \Pi(0) \text{ and } b = \Pi(1/2) \text{ are fixed points of } f. \text{ If } x \text{ is not } a \text{ and } b \text{ in } S^1, \text{ then } \lim_{n \to \infty} f^n(x) = b. \text{ By } [4], \text{ } f \text{ has the shadowing property. But the point } \Pi(1/2) \text{ is not chain recurrent point, and by the theorem 1.1 in } [6], \text{ } f \text{ does not satisfy the average shadowing property.} \]

We use the theorem 1.1 in [6] to drive another characterization of the average shadowing property.

**Theorem 3.4.** If a continuous surjective map \( f \) on a compact metric space \( X \) has the average shadowing property, then every point \( x \) is chain recurrent. Moreover \( f \) has only one chain component which is the whole space.

**Proof.** Let \((X, d)\) be a compact metric space and let \( f : X \to X \) be a continuous surjective map with the average shadowing property. It is sufficient to prove that for any two different points \( x, y \in X \), \( x \not\to y \).

Let \( x, y \) be any two different points of \( X \). We denote \( D \) by the diameter of \( X \), that is, \( D = \max_{(x, y) \in X \times X} d(x, y) \). If \( y \) is in the positive orbit of \( X \), then \( x \not\to y \). So we assume that \( y \) is not in the positive orbit of \( x \). For any \( \epsilon > 0 \), take \( 0 < \epsilon \leq 4/9 \) such that if \( d(x, y) < 2\epsilon \), then \( d(f(x), f(y)) < \epsilon \). Let \( \delta = 3\epsilon > 0 \) be a number as in the definition of the average shadowing property \( f \), that is, every \( \delta \)-average-pseudo orbit \( \{x_i\}_{i=0}^{\infty} \) is \( \epsilon \)-shadowed in average by some \( z \) in \( X \). Fix a sufficient large integer \( N_0 > 0 \) which \( \frac{3D}{N_0} < \delta \).

Consider a subset \( S_1 \) of \( X \) which satisfy \( f(S_1) = \{y\} \). Take a point \( y_1 \in S_1 \). Then \( f(y_1) = y \), for \( y_1 \in S \). Again we consider a subset \( S_i \) of
X and take a point $y_i$ in $S_i$ satisfying
\[ f(S_i) = \{y_{i-1}\}, \quad 1 < i \leq N_0 - 2. \]

Define a cyclic sequence $\{x_i\}_{i=0}^{\infty}$ by
\[
\begin{cases}
  x_i = f[i \mod 2N_0] & \text{if } [i \mod 2N_0] \in [0, N_0] \\
  x_i = y_{2N_0 - ([i \mod 2N_0] + 1)} & \text{if } [i \mod 2N_0] \in [N_0 + 1, 2N_0 - 2] \\
  x_i = y & \text{if } [i \mod 2N_0] = 2N_0 - 1.
\end{cases}
\]

Then for $n \geq N_0$ and $k > 0$,
\[
\frac{1}{n} \sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \frac{1}{n} \cdot \frac{n}{N_0} \cdot 3D \leq \frac{3D}{N_0} < \delta.
\]

Thus $\{x_i\}_{i=0}^{\infty}$ is a cyclic $\delta$-average-pseudo-orbit of $f$. Hence it is $\epsilon$-shadowed in average by some $z \in X$, that is,
\[
\lim_{n \to \infty} \sup \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) < \epsilon.
\]

Put $P_1 = \{x, f(x), \ldots, f^{N_0}(x)\}$. Then we have a following result.

Claim: There exists an infinite sequence $\{i_1, i_2, \ldots\}$, $i_s < i_k$ for $s < k$ such that
\[ B(f^{i_s}(z), 2\epsilon) \cap P_1 \neq \emptyset \quad \text{and} \quad d(f^{i_s}(z), x_{i_s}) < 2\epsilon \quad \text{for all } i_j \in \{i_1, i_2, \ldots\}. \]

Otherwise, there exists a natural number $N$ such that for all $i > N$,
\[ d(f^i(z), x_i) \geq 2\epsilon. \]

Then
\[
\lim_{n \to \infty} \sup \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) \geq 2\epsilon,
\]

which is a contradiction. Put $P_2 = \{y_{N_0-2}, \ldots, y_1, y\}$. Then similar result hold for $P_2$. There is an infinite sequence $\{l_1, l_2, \ldots\}$, $l_s < l_k$ if $s < k$ such that
\[ B(f^{l_s}(z), 2\epsilon) \cap P_2 \neq \emptyset \quad \text{and} \quad d(f^{l_s}(z), x_{l_s}) < 2\epsilon. \]

Now choose
\[ i_0 \in \{i_1, i_2, \ldots\} \quad \text{and} \quad l_0 \in \{l_1, l_2, \ldots\} \quad \text{with} \quad i_0 < l_0 \]

such that $x_{i_0} \in P_1$ and $x_{l_0} \in P_2$. Then
\[ d(f^{i_0}(z), x_{i_0}) < 2\epsilon \quad \text{and} \quad d(f^{l_0}(z), x_{l_0}) < 2\epsilon. \]
By assuming
\[
\begin{cases}
x_{i_0} = f^{j_1}(x) & \text{for some } j_1 > 0 \\
x_{i_0} = y_{j_2} & \text{for some } j_2 > 0,
\end{cases}
\]
we have the following an \( \epsilon_0 \)-pseudo-orbit from \( x \) to \( y \)
\[
x, \ f(x), \ldots, f^{j_1}(x) = x_{i_0}, \\
f^{i_0+1}(z), \ f^{i_0+2}(z), \ldots, \ f^{i_0-1}(z) \\
x_{i_0} = y_{j_2}, \ y_{j_2-1}, \ldots, \ y.
\]
This proves \( x \xrightarrow{f} y \) and we complete the proof of Theorem 3.4. \( \square \)

References


Jong-Jin Park
Department of Mathematics
Chonbuk National University
Chonju 561-756, Korea
E-mail: jjpark46@chonbuk.ac.kr

Yong Zhang
Department of Mathematics
Suzhou University
Suzhou, China
E-mail: yongzhang@suda.edu.cn