GENERALIZED TOEPLITZ ALGEBRA OF A CERTAIN NON-AMENABLE SEMIGROUP

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Abstract. We analyze a detailed picture of the algebraic structure of C*-algebras generated by isometric representations of the non-amenable semigroup $P = \{0, 2, 3, \ldots, n, \ldots\}$.

1. Introduction

Let $S$ denote a countable discrete semigroup with unit $e$ and $B$ be a unital $C^*$-algebra. A map $W : S \rightarrow B, x \mapsto W_x$ is called an isometric homomorphism if $W_e = 1$, $W_x$ is an isometry and $W_{xy} = W_xW_y$ for all $x, y \in S$. If $B$ is the $C^*$-algebra $B(H)$ of all bounded linear operators of a non-zero Hilbert space $H$, we call $(H, W)$ an isometric representation of $S$.

If $S$ is left-cancellative, then we can have a specific isometric representation of $S$, called the left regular isometric representation on the Hilbert space $l^2(S)$. The left regular isometric representation $L : S \rightarrow B(l^2(S)), x \mapsto L_x$ is defined by the equation

$$(L_xf)(z) = \begin{cases} f(y), & \text{if } z = xy \text{ for some } y \in M, \\ 0, & \text{if } z \notin xM. \end{cases}$$

In fact, when $\{\delta_x \mid x \in S\}$ is the canonical orthonormal basis of the Hilbert space $l^2(S)$ defined by

$$\delta_x(y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{otherwise}, \end{cases}$$

Received February 14, 2005.
2000 Mathematics Subject Classification: 46L05, 47C15, 47B35.
Key words and phrases: isometric homomorphism, left regular isometric representation, reduced semigroup $C^*$-algebra, semigroup $C^*$-algebra, Toeplitz algebra.
This paper is partially supported by 2004 Research Fund of University of Ulsan.
we have that $L_x(\delta_y) = \delta_{xy}$ for all $x, y \in S$.

In order to make things explicit, let us consider the semigroup $\mathbb{N}$ of all natural numbers. The isometry $L_1$ of the left regular isometric representation $L: \mathbb{N} \to \mathcal{B}(l^2(\mathbb{N})), x \mapsto L_x$ is the unilateral shift of $l^2(\mathbb{N})$ on the canonical orthonormal basis $\{\delta_n \mid n \in \mathbb{N}\}$.

Among $C^*$-algebras generated by isometries, the $C^*$-algebra generated by the left regular isometric representation of a left cancellative semigroup can be considered as the appropriate analogue for the Toeplitz algebra. The $C^*$-algebra generated by the left regular isometric representation of a left cancellative semigroup $S$ has several names. We shall call it the reduced semigroup $C^*$-algebra, and denote it $C^*_\text{red}(S)$ as in the paper [5].

Besides the reduced semigroup $C^*$-algebra, we will consider the semigroup $C^*$-algebra introduced by G. J. Murphy [8]. The semigroup $C^*$-algebra is obtained by enveloping all isometric representations of $S$, and is denoted by $C^*(S)$. From the construction of $C^*(S)$ we have naturally a canonical isometric homomorphism $V$ from $S$ to $C^*(S)$. It follows from the definition of the semigroup $C^*$-algebra it has the following universal property: If $V$ is the canonical isometric homomorphism from $S$ into the semigroup $C^*$-algebra $C^*(S)$, then for any isometric homomorphism $W$ from $S$ into a unital $C^*$-algebra $B$ there exists a unique homomorphism from the semigroup $C^*$-algebra $C^*(S)$ into the unital $C^*$-algebra $B$ sending a canonical isometry $V_x$ to an isometry $W_x$ for each $x \in S$.

Ever since L. A. Coburn proved his well-known theorem, which asserts that the $C^*$-algebra generated by a non-unitary isometry on a separable infinite dimensional Hilbert space does not depend on the particular choice of the isometry [1], many authors have taken an interest in the generalization of Coburn’s theorem. It is sometimes called the uniqueness property of the $C^*$-algebras generated by isometries. If the $C^*$-algebras generated by isometries have the uniqueness property, the structures of those $C^*$-algebras are to some extent independent of the choice of isometries on a Hilbert space. The uniqueness property of $C^*$-algebras generated by isometries describes when the reduced semigroup $C^*$-algebra $C^*_\text{red}(S)$ and the semigroup $C^*$-algebra $C^*(S)$ are isomorphic or when the reduced semigroup $C^*$-algebra $C^*_\text{red}(S)$ has a universal property for certain kinds of isometric representations of $S$ [2, 3, 4, 6, 7]. As good examples of the uniqueness property, we note that all the $C^*$-algebras generated by the isometric representations of the semigroup $\mathbb{N}$ of all natural numbers are isomorphic to the Toeplitz algebra. The $C^*$-algebras generated by one parameter semigroup of isometries and
the Cuntz algebras are also remarkable examples of the $C^*$-algebras of isometries which have the uniqueness property.

A. Nica introduced the quasi-lattice group $(G, S)$, the covariant isometry representations of semigroups and the amenability problem of quasi lattice ordered groups in order to find the condition that the reduced semigroup $C^*$-algebra $C^*_{red}(S)$ has a universal property for certain kinds of isometric representations of $S$ [9]. The partially ordered group $(G, S)$ is quasi-lattice ordered group if every finite subset of $G$ with an upper bound in $S$ has a least upper bound in $S$. The amenability problem, which asks when the left regular isometric representations have the universal property of the covariant isometric representations, was also investigated in [6]. The quasi-lattice ordered group is an appropriate concept for the universal property of the reduced semigroup $C^*$-algebras.

In this paper we show that the uniqueness property is much dependent on the order structure of the semigroup $S$, by analyzing the structure of the reduced semigroup $C^*$-algebra $C^*_{red}(P)$ of $P$ and the semigroup $C^*$-algebra $C^*(P)$ of $P$, where $P = \{0, 2, 3, \ldots\}$.

The semigroup $P = \{0, 2, 3, \ldots\}$ is a generating subsemigroup of the integer group $\mathbb{Z}$. By Coburn’s result it is known that the reduced semigroup $C^*$-algebra $C^*(\mathbb{N})$ of $\mathbb{N}$ is isomorphic to the semigroup $C^*(\mathbb{N})$ which is isomorphic to the Toeplitz algebra. We show that the reduced semigroup $C^*$-algebra $C^*_{red}(P)$ is isomorphic to the Toeplitz algebra $C^*(\mathbb{N})$, but we also show that $C^*_{red}(P)$ is not isomorphic to $C^*(P)$ by using the order structure of $P$ in Proposition 2.7. Our semigroup $P$ is abelian and really simple one, but not quasi lattice ordered.

2. Main result

Let $G$ be a countable discrete group and $S$ a subsemigroup of $G$ with the unit $e$. We define an order on $G$ as follows: Two elements $x$ and $y$ in $S$ are comparable when $x \in yS$ or $y \in xS$. If $x$ is contained in $yS$, then $x$ is larger than $y$ and we denote it by $y \leq x$. This relation makes $(G, S)$ a pre-ordered group. If the unit $e$ of $S$ is the only invertible element of $S$, $(G, S, \leq)$ is a partially ordered group.

We can identify a maximal and a minimal element in $S$ in the following sense: an element $x_0 \in S$ is maximal if and only if $x_0 \leq x$ implies that $x = x_0$ and an element $x_1$ is minimal if and only if $x \leq x_1$ implies that $x_1 = x$ for $x \in S$.

The reduced semigroup $C^*$-algebra $C^*(S)$ is generated by $\{L_x \mid x \in S\}$. The partial order $\leq$ on $S$ is transitive: if $x \leq y \leq z$, then $x \leq z$.
\[ S \}, \text{ where } L \text{ is the left regular isometric representation of } S. \text{ In fact, } C^*(S) \text{ is the closed linear span of } \{ L_{x_1} L_{x_2}^* \cdots L_{x_{2k}}^* L_{x_{2k+1}} \mid x_i \in S \}. \text{ If the semigroup } S \text{ is the semigroup } \mathbb{N} \text{ of natural numbers, then } C^*(\mathbb{N}) \text{ is the Toeplitz algebra. So sometimes the reduced semigroup } C^*\text{-algebra } C^*(S) \text{ is called a generalized Toeplitz algebra.}

**Proposition 2.1.** If the unit of } S \text{ is the only invertible element of } S, \text{ then } \{ L_x L_y^* \mid x, y \in S \} \text{ is linearly independent.}

**Proof.** First, we can see that \( L_x L_y^* (\delta_y) = \delta_x \) for all } x, y \in S, \text{ so } L_x L_y^* \text{ never can be zero for any } x, y \in S.

Suppose that there exist } \{ \lambda_i \mid \lambda_i \in \mathbb{C}, 1 \leq i \leq n \} \text{ and } \{(x_i, y_i) \mid x_i, y_i \in S, (x_i, y_i) \neq (x_j, y_j) \text{ for } i \neq j, 1 \leq i, j \leq n \} \text{ such that}

\[
\sum_{i=1}^{n} \lambda_i L_{x_i} L_{y_i}^* = 0.
\]

We can divide } \{ y_i \mid 1 \leq i \leq n \} \text{ into two kinds of subsets; the one consists of elements which are comparable with any other element of } \{ y_i \mid 1 \leq i \leq n \} \text{ and the other is the rest.}

Let } y_{i_0} \text{ be the element of } \{ y_i \mid 1 \leq i \leq n \} \text{ which is not comparable with any other element of } \{ y_i \mid 1 \leq i \leq n \}. \text{ Since } L_x, L_{y_{i_0}} (\delta_{y_{i_0}}) = \delta_x, \text{ for } y_i = y_{i_0}, \text{ we can have}

\[
\sum_{i=1}^{n} \lambda_i L_{x_i} L_{y_{i_0}}^* (\delta_{y_{i_0}}) = \lambda_{i_1} \delta_{x_{i_1}} + \cdots + \lambda_{i_k} \delta_{x_{i_k}} = 0,
\]

if } y_{i_j} = y_{i_{i_0}} \text{ for } j = 1, \ldots, k \text{ and } \{ l_1, \ldots, l_k \} \subset \{ 1, 2, \ldots, n \}. \text{ Since } x_{i_r} \neq x_{i_s} \text{ if } l_r \neq l_s \text{ for } 1 \leq r, s \leq k, \text{ we have } \lambda_{i_j} = 0 \text{ for } j = i, \ldots, k.

Next, since } \{ y_{i_1}, \ldots, y_{i_n} \} \text{ is finite, we can consider a minimal element } y_{i_1} \text{ of some chain of } \{ y_{i_1}, \ldots, y_{i_n} \}. \text{ If we look at prudently the term}

\[
\sum \lambda_i L_{x_i} L_{y_{i_1}}^* (\delta_{y_{i_1}}),
\]

we can see that only the terms with } y_{i_1} \text{ may not be zero. So we have}

\[
\sum_{i=1}^{n} \lambda_i L_{x_i} L_{y_{i_1}}^* (\delta_{y_{i_1}}) = \lambda_{m_1} \delta_{x_{m_1}} + \cdots + \lambda_{m_p} \delta_{x_{m_p}} = 0,
\]

if } y_{m_j} = y_{i_1} \text{ for } j = 1, \ldots, p \text{ and } \{ m_1, m_2, \ldots, m_p \} \subset \{ 1, 2, \ldots, n \}.

By the similar computation as the above, we can see that } \lambda_{m_j} = 0 \text{ for } j = 1, \ldots, p. \text{ So we can exclude those terms in the two cases and have a
is the strong closure of $B$. Furthermore, if we put $\delta$ semigroups. not be abelian, and are moreover primitive for a large class of abelian.

The group $C^*$-algebra of an abelian group is, of course, itself abelian and so not very interesting from the point of view of $C^*$-theory. But the reduced semigroup $C^*$-algebra and the semigroup $C^*$-algebra may not be abelian, and are moreover primitive for a large class of abelian semigroups.
Proposition 2.4. The commutator ideal $Z(C^*_{red}(P))$ of $C^*_{red}(P)$ is the algebra $K(l^2(P))$ of compact operators on the Hilbert space $l^2(P)$.

Proof. Since $C^*_{red}(P)$ is generated by $L_2$ and $L_3$, it is enough to see how these operators act on $l^2(P)$. The operator $I - L_2L_2^*$ is of finite rank, and so is contained in $K(l^2(P))$. Therefore, $K(l^2(P))$ and the commutator ideal $Z(C^*_{red}(P))$ have non-empty intersection. Since $C^*_{red}(P)$ acts irreducibly on $l^2(P)$, the commutator ideal $Z(C^*_{red}(P))$ contains the algebra $K(l^2(P))$ of compact operators.\[\square\]

Though there are many interesting simple group $C^*$-algebras, the reduced semigroup $C^*$-algebras are rarely simple for a large and natural class of semigroups. In fact, there are many prime reduced semigroup $C^*$-algebras and it is still open when the reduced semigroup $C^*$-algebra is prime. We can see that $C^*_{red}(P)$ is prime from the Proposition 2.4.

The semigroup $P$ has two generators, so apparently $C^*_{red}(P)$ is generated by two non-unitary isometries. However, the following theorem shows us that $C^*_{red}(P)$ is generated by a non-unitary isometry.

Theorem 2.5. The $C^*$-algebra $C^*_{red}(P)$ is generated by a single non-unitary isometry.

Proof. The operator $L_2^*L_3$ acts on $l^2(P)$ as follows:

$$L_2^*L_3(\delta_n) = \begin{cases} 0, & \text{if } n = 0, \\ \delta_{n+1}, & \text{if } n \neq 0. \end{cases}$$

Hence $L_2^*L_3$ translates every element of the orthonormal basis $\{\delta_0, \delta_2, \delta_3, \ldots\}$ of $l^2(P)$ except $\delta_0$. Let $K$ be the compact operator defined by

$$K(\delta_n) = \begin{cases} \delta_2, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

Put $U = L_2^*L_3 + K$. By Proposition 2.3, $U$ is contained in $C^*_{red}(P)$.

$U^*U$ is the identity operator on $l^2(P)$ because $L_2^*L_2L_2^*L_3$ is the projection onto the closed subspace spanned by $\{\delta_n \mid n \in P, n \neq 0\}$.

$K^*K$ is the projection onto the closed subspace $C\delta_0$ and all other operators in the terms of $U^*U$ are zero. Similarly, we see that $UU^*$ is the projection onto the closed subspace spanned by $\{\delta_2, \delta_3, \ldots\}$. In fact the operator
$U$ sends $\delta_0$ to $\delta_2$ and $\delta_n$ to $\delta_{n+1}$ for $n \neq 0$. So the operator $U$ is the unilateral shift on $l^2(P)$ with respect to the canonical orthonormal basis $\{\delta_n \mid n \in P\}$. Let $T$ be the $C^*$-subalgebra of $C^*_\text{red}(P)$ generated by $U$. If we consider the compact operators $T_1$ and $T_2$ defined by

$T_1(\delta_n) = \begin{cases} -\delta_3, & \text{if } n = 0, \\ 0, & \text{otherwise,} \end{cases}$ and $T_2(\delta_n) = \begin{cases} \delta_2, & \text{if } n = 0, \\ 0, & \text{otherwise,} \end{cases}$

then we can show that $L_2 = U^2 + T_1 + T_2$ because $U^2 + T_1 + T_2(\delta_n) = \delta_{n+2}$ for each $n \in P$. Similarly we can show that $L_3 = U^3 + T_3 + T_4$ where

$T_3(\delta_n) = \begin{cases} -\delta_4, & \text{if } n = 0, \\ 0, & \text{otherwise,} \end{cases}$ and $T_4(\delta_n) = \begin{cases} \delta_3, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$

Since the algebra $T$ contains the compact operator algebra $K(l^2(P))$, the elements $U^2 + T_1 + T_2$ and $U^3 + T_3 + T_4$ are contained in $T$. Hence we conclude that $C^*_\text{red}(P)$ is same as the algebra $T$ because $C^*_\text{red}(P)$ is generated by $L_2$ and $L_3$. □

**Corollary 2.6.** $C^*_\text{red}(P)$ is isomorphic to the Toeplitz algebra.

From the point of view of $C^*$-algebras, amenability means that the canonical coincidence of two kinds $C^*$-algebras: One is the universal object obtained by enveloping a certain class of representation and the other is associated to a specific representations of the class.

A. Nica introduced the quasi-lattice ordered group $(G, S)$, the covariant isometric representations of semigroups and the amenability problem of quasi-lattice ordered groups for the universal property of the reduced semigroup $C^*$-algebra $C^*_\text{red}(S)$.

The partially ordered group $(G, S)$ is said to be quasi-lattice ordered if the following is satisfied: If $x_1, x_2, \ldots, x_n$ in $G$ which have common upper bounds in $S$ for any $n \geq 1$, they also have a least common upper bound in $S$.

The above condition can be expressed in another form consisting of two conditions:

1. Any $x$ in $SS^{-1}$ has a least upper bound in $S$.
2. Any $s, t$ in $S$ with the common upper bound has a least common upper bound.

If $(G, S)$ is a quasi-lattice ordered group and $x_1, x_2, \ldots, x_n$ in $G$ have a common upper bound in $S$, then their least common upper bound in $S$ will be denoted by $\sigma(x_1, x_2, \ldots, x_n)$. 

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An isometric representation $V$ of $S$ on the Hilbert space $H$ is said to be **covariant** if

$$V(s)V(t) = \begin{cases} V(\sigma(s,t)), & \text{if } s \text{ and } t \text{ have common upper bound} \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that for the semigroup $S$ of the quasi-lattice ordered group $(G, S)$ the left regular isometric representation $\mathcal{L}$ of the semigroup $S$ is a covariant representation. The reduced semigroup $C^*$-algebra $C^*_{red}(S)$ naturally plays the role of reduced $C^*$-algebra in the class of $C^*$-algebras of covariance isometric representations. Nica also constructed the full $C^*$-algebra $C^*_{cov}(G,S)$ of the $C^*$-algebras generated by enveloping covariant representations of $(G, S)$ [9].

The quasi-lattice ordered group $(G, S)$ is amenable if the reduced semigroup $C^*$-algebra $C^*_{red}(S)$ is isomorphic to $C^*_{cov}(G, S)$. It is known that every abelian quasi-lattice ordered group is amenable. Our semigroup $P$ is very simple and abelian. But it is not quasi-lattice ordered, because 2 and 3 in the semigroup $P = \{0, 2, 3, 4, \ldots\}$ have common upper bounds 5 and 6. Since 5 and 6 are not comparable, 2 and 3 does not have a least common upper bound.

**Proposition 2.7.** The reduced semigroup $C^*$-algebra $C^*_{red}(P)$ is not isomorphic to the semigroup $C^*$-algebra $C^*(P)$.

**Proof.** The left regular isometric representation $\mathcal{L}$ satisfies the relation:

$$\mathcal{L}_2 \mathcal{L}_3 (I - \mathcal{L}_2 \mathcal{L}_2^*)(I - \mathcal{L}_3 \mathcal{L}_3^*) = 0. \tag{1}$$

Let $W$ be the isometric representation of $P$ defined by $W_n = S^n$ for $n = 0, 2, 3, \ldots$, where $S$ is the unilateral shift on $l^2(\mathbb{N})$. This representation does not satisfy the above relation, i.e.,

$$S^{s^2}S^3(I - S^2S^{s^2})(I - S^3S^{s^3}) \neq 0. \tag{2}$$

Let $\mathcal{W}$ be the $C^*$-algebra generated by the isometric representation $W$ of $P$. Since $C^*(P)$ has the universal property, there is a homomorphism from $C^*(P)$ to $\mathcal{W}$ sending $V_n$ to $W_n$ for each $n \in P$. But there does not exist a homomorphism from $C^*_{red}(P)$ to $\mathcal{W}$ sending $\mathcal{L}_n$ to $W_n$ for each $n \in P$ because of equations (1) and (2). So the reduced semigroup $C^*$-algebra $C^*_{red}(P)$ is not isomorphic to the semigroup $C^*$-algebra $C^*(P)$. \[\square\]
References


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