RIBAUCOUR TRANSFORMATION FOR FLAT $m$-SUBMANIFOLDS IN $\mathbb{H}^{m+n}$

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Abstract. By using $O(m + n, 1)/O(m) \times O(n, 1)$-system, we give an analytic version of Ribaucour transformation for flat $m$-dimensional submanifolds in $\mathbb{H}^{m+n}$ with flat, non-degenerate normal bundle and free of weak umbilics, where $n \geq m - 1$.

1. Introduction

The study of immersions of space forms into space forms is one of the most important and interesting problems in classical differential geometry. E. Cartan showed that an $n$-dimensional hyperbolic space form can be locally immersed in $\mathbb{H}^{2n-1}$ and the dimension $2n - 1$ cannot be lowered [2, 6]. It is a classical result due to Hilbert [5] that there are no complete isometric immersions $N^2(c') \rightarrow N^3(c)$ if $c' < c$ and $c' < 0$, but it is yet unknown (though conjectured) whether this result extends to complete isometric immersions $N^n(c') \rightarrow N^{2n-1}(c)$ for $c' < c$ and $c' < 0$. Notice that for the case $c' = 0$, one always has the Clifford tori, and $c' > 0$ cannot occur due to the fact that such immersions induce global Chebyshev coordinates [7, 11]. In contrast, when $c' > c$, one always has the totally umbilical hypersurfaces. Especially if the immersion has no umbilic points, then the normal bundle is flat [7]. Later, in [13, 14], Tenenhabl and Terng studied immersions of $\mathbb{H}^n$ into $\mathbb{E}^{2n-1}$ and $\mathbb{E}^n$ into $\mathbb{S}^{2n-1}(1)$, too.

Recently Terng et al. in [12, 1] showed that to each symmetric space $G/K$ one can associate to an $m$-dimensional first order systems of partial differential equations (PDEs in brief), the so-called $G/K$-system. By using $G_{pq}^{m,n}$-systems they studied local non-degenerate isometric immersions $X: N^m_s(c') \rightarrow N^{m+k}_s(c)$ ($s = 0$ or 1) with flat normal bundle and...
c′ ≤ c, where $G_{m,n}^{p,q} = O(m + n, p + q)/O(m, p) \times O(n, q)$. On the other hand, the classical Ribaucour transformation for surfaces in Euclidean 3-space was extended for submanifolds of space forms with arbitrary dimension and codimension in [4], where it was applied to develop a transformation theory for submanifolds with constant sectional curvature. It was later shown that the dressing action of a simple element on a solution of the $G_{n,m}^{p,q}$-system corresponds to a Ribaucour transformation of the associated submanifold, see [12, 1, 8, 9, 10, 15] for details.

A natural question is whether Terng’s method can be used to study local isometric immersions $X : \mathbb{N}_{m}(c') \rightarrow \mathbb{N}_{m+n}(c)$ with flat normal bundle and $c' > c$. In this paper we shall discuss $c' = 0$, $c = -1$ and $n \geq m - 1$ and obtain an analogous results of [1] for flat $m$-dimensional submanifolds in $\mathbb{S}^{2n-1}$. We notice that the Gauss-Codazzi equations of $X$ under certain conditions are gauge equivalent to the $G_{m,n}^{1} = O(m + n, 1)/O(m) \times O(n, 1)$-system and show that the dressing action of a simple element on the space of solutions of the $G_{m,n}^{1}$-system gives rise to a Ribaucour transformation of the flat $m$-dimensional submanifold $X$ in $\mathbb{H}^{m+n}$.

2. The general cases of the $G_{n,m}^{1}$-system

G/K-systems were introduced for a symmetric space G/K by Terng et al in [12, 1]. To keep self-contained, below we give a short review of some known facts about the $G_{m,n}^{1}$-system, see [1, 15] for details.

Definition 2.1. ([1, 12]) The $G_{m,n}^{1}$-system is the following PDEs for $\xi = (\xi_{ij}) : R^{m} \rightarrow \mathcal{M}_{m \times (n+1)}$ with $\xi_{ii} = 0$ for all $1 \leq i \leq m$ such that

\begin{equation}
\theta_{\lambda} = \sum_{i=1}^{m} \left( \frac{D_{i} \xi - \xi D_{i}^{t}}{\lambda D_{i}^{t}} - \frac{\lambda D_{i} I_{n,1}}{D_{i}^{t} \xi - I_{n,1} \xi^{t} D_{i} I_{n,1}} \right) dx_{i}
\end{equation}

is a family of flat connections on $R^{m}$ for all $\lambda \in \mathbb{C}$, i.e.,

\begin{equation}
d\theta_{\lambda} + \theta_{\lambda} \wedge \theta_{\lambda} = 0.
\end{equation}

Here $D_{i} \in \mathcal{M}_{m \times (n+1)}$ is the matrix all whose entries are zero except that $(i, i)$-th entry is equal to 1.

It follows from (2.2) that for a given initial value $E(0, \lambda)$ there exists a smooth map $E : \mathbb{R}^{m} \times \mathbb{C} \rightarrow O(m + n, 1)$, called the frame of $\theta_{\lambda}$, such
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that $E^{-1}dE = \theta_\lambda$. The $G_{m,n}^1$-reality condition is

$$
(2.3) \quad \begin{cases}
  g(\lambda) = g(\lambda), \\
  I_{m,n+1}g(\lambda)I_{m,n+1} = g(-\lambda), \\
  g(\lambda)I_{m+n,1}g(\lambda)^t = I_{m+n,1}.
\end{cases}
$$

Let $g = \left( \begin{array}{cc} B & 0 \\ 0 & A \end{array} \right) \in O(m) \times O(n,1)$ be a solution of $g^{-1}dg = \theta_0$.

Write

$$
\xi = (F, G), \quad D_1 = (C_i, 0), \quad A = (A_1, A_2), \quad F, C \in gl(m), \quad G \in \mathcal{M}_{m \times (n+1-m)}, \quad A_1 \in \mathcal{M}_{(n+1) \times m}, \quad A_2 \in \mathcal{M}_{(n+1) \times (n+1-m)},
$$

and

$$
h = \left( \begin{array}{cc} I_m & 0 \\ 0 & A \end{array} \right).
$$

To do the gauge transformation of $\theta_\lambda$ of by $h$, we have

$$
(2.4) \quad \Omega_\lambda = h \ast \theta_\lambda = h\theta_\lambda h^{-1} - dhh^{-1} = \sum_{i=1}^m \left( C_i F^t - FC_i^t \right) \frac{-C_i A_1^{t}I_{n,1} \lambda}{A_1 C_i^t \lambda} dx_i.
$$

It is also a family of flat connections on $\mathbb{R}^n$ for all $\lambda \in \mathbb{C}$, i.e.,

$$
(2.5) \quad \begin{cases}
  (f_{ij})_{x_i} + (f_{ji})_{x_j} + \sum_{k=1}^m f_{ki}f_{kj} = 0, & \text{if } i \neq j, \\
  (f_{ij})_{x_k} = f_{ik}f_{kj}, & \text{if } i, j, k \text{ are distinct}, \\
  (a_{ij})_{x_k} = a_{ik}f_{kj}, & \text{if } j \neq k,
\end{cases}
$$

where $A = (a_{ij}) \in O(n,1)$ and $F = (f_{ij}) \in gl(m) = \{(g_{ij}) \in gl(m) | g_{ii} = 0, \ 1 \leq i \leq m\}$. Note that $Eh^{-1}$ is a frame of $\Omega_\lambda$. The system (2.5) is called the $G_{m,n}^1$-system.

In [15] we have obtained an explicit construction of the dressing action of a rational map with two simple poles of solutions of the $G_{m,n}^{p,q}$-system. Here we only state the result. Let $\mathbb{C}^{m+n+1}$ be equipped with the bilinear form

$$
\langle u, v \rangle_1 = \sum_{i=1}^{m+n} \bar{u}_i v_i - \bar{u}_{m+n+1} v_{m+n+1}.
$$

Let $W$ and $Z$ be unit space-like constant vectors in $\mathbb{R}^m, \mathbb{R}^{n,1}$ respectively, and $\pi$ the orthogonal projection onto the space of $\mathbb{C} \left( \begin{array}{c} W \\ iZ \end{array} \right)$ with respect to $\langle \ , \ \rangle_1$. Define

$$
(2.6) \quad g_{s,\pi} = \left( \pi + \frac{\lambda - is}{\lambda + is} (I - \pi) \right) \left( \bar{\pi} + \frac{\lambda + is}{\lambda - is} (I - \bar{\pi}) \right), \quad 0 \neq s \in \mathbb{R}.
$$
Lemma 2.2. ([15]) Let $\xi : \mathbb{R}^m \to \mathcal{M}_{m \times (n+1)}$ be a solution of the $G^1_{m,n}$ system (2.2), and $E(x, \lambda)$ a frame of $\xi$ such that $E(x, \lambda)$ is holomorphic for $\lambda \in \mathbb{C}$ and $E(0, \lambda) = I$. Let $g_{s, \pi}$ as in (2.6) and

$$
(2.7) \quad \left( \begin{array}{c} \tilde{W} \\ i \tilde{Z} \end{array} \right)(x) = E(x, -is)^{-1} \left( \begin{array}{c} W \\ iZ \end{array} \right),
$$

and $\hat{\pi}$ the orthogonal projection onto the space of $\mathbb{C} \left( \begin{array}{c} \tilde{W} \\ i \tilde{Z} \end{array} \right)$ with respect to $\langle , \rangle_1$. Then $\tilde{\xi} = g_{s, \pi} \xi = \xi - 2s(\tilde{W} \tilde{Z})^t I_{n,1}$ is a new solution of (2.2) and $\tilde{E} = E g_{s, \pi}$ is a frame for $\tilde{\xi}$, where $\tilde{W} = \frac{W}{\|W\|_{Rm}}$ and $\tilde{Z} = \frac{Z}{\|Z\|_{Rn,1}}$.

Note that both $E$ and $\tilde{E}$ satisfy the $G^1_{m,n}$-reality conditions (2.3) which implies that $E(x, 0)$ and $\tilde{E}(x, 0)$ are in $O(m + n, 1)$. Write

$$
E(x, 0) = \left( \begin{array}{cc} B(x) & 0 \\ 0 & A(x) \end{array} \right), \quad \tilde{E}(x, 0) = \left( \begin{array}{cc} \tilde{B}(x) & 0 \\ 0 & \tilde{A}(x) \end{array} \right)
$$

for some $A, B, \tilde{A}$ and $\tilde{B}$, and we have

$$
\begin{align*}
\tilde{A} &= A(I - 2\tilde{Z} \tilde{Z}^t I_{n,1}), \\
\tilde{B} &= B(I - 2\tilde{W} \tilde{W}^t).
\end{align*}
$$

Write

$$
\xi = (F, G), \quad \tilde{\xi} = (\tilde{F}, \tilde{G}), \quad A = (A_1, A_2), \quad \tilde{A} = (\tilde{A}_1, \tilde{A}_2).
$$

Here $A_1, \tilde{A}_1 \in \mathcal{M}_{(n+1) \times m}$, and $A_2, \tilde{A}_2 \in \mathcal{M}_{(n+1) \times (n+1-m)}$. Obviously $(A_1, F)$ and $(\tilde{A}_1, \tilde{F})$ are solutions of the $G^1_{m,n}$-II system (2.5), the corresponding frames are $E^{II}(x, \lambda)$ and $\tilde{E}^{II}(x, \lambda)$, where

$$
E^{II}(x, \lambda) = E(x, \lambda) \left( \begin{array}{cc} I_m & 0 \\ 0 & A^{-1} \end{array} \right), \quad \tilde{E}^{II}(x, \lambda) = \tilde{E}(x, \lambda) \left( \begin{array}{cc} I_m & 0 \\ 0 & \tilde{A}^{-1} \end{array} \right).
$$

It follows from lemma 2.2 that

$$
\tilde{E}^{II}(x, \lambda) = E^{II}(x, \lambda) \left( I - \frac{2}{s^2 + x^2} \begin{bmatrix} s \tilde{W} \\ -s \lambda \tilde{A} \tilde{Z} \end{bmatrix} \right) \left( I - \frac{2}{s^2 + x^2} \begin{bmatrix} \lambda \tilde{Z}^t \\ -s \lambda \tilde{W}^t \end{bmatrix} \right).
$$

In the following, we use the notation:

$$
(\tilde{A}_1, \tilde{F}, \tilde{E}^{II}) = g_{s, \pi}((A_1, F, E^{II})�.
$$
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In this section we will show that the dressing action of $g_{s,\pi}$ on the spaces of solutions of the $G_{m,n}^1$-II system (2.2) gives rise to a Ribaucour transformation for flat $m$-dimensional submanifolds with flat and non-degenerate normal bundle in $\mathbb{H}^{m+n}$, where $n \geq m - 1$.

Let $X$ be a flat $m$-dimensional submanifold in $\mathbb{H}^{m+n}$ with flat, non-degenerate normal bundle and free of weak umbilics. Recall from [7, 3] that a point $x \in X$ is said to be weak umbilics if there is a unit normal vector $\zeta$ at $x$ such that $A_\zeta = I_d$, where $A_\zeta$ denotes the shape operator in the direction $\zeta$. Fix a local parallel normal frame $\{e_{m+1}, \ldots, e_{m+n}\}$, then there exist (see [3] for details) line of curvature coordinates $\{x_1, \ldots, x_m\}$ and a smooth map $A_1 \in \mathcal{M}_{(n+1)\times m}$ such that

$$A_1^t I_{n,1} A_1 = \begin{cases} I_m & \text{if } n \geq m \\ I_{m-1,1} & \text{if } n = m - 1 \end{cases}$$

and the first and the second fundamental forms are

$$ds^2 = \sum_{k=1}^m a_{n+1,k}^2 dx_k^2, \quad II = \sum_{k=1}^m \sum_{j=1}^n a_{n+1,k} a_{jk} dx_k^2 e_{m+j}.$$  

A direct computation gives the following proposition.

**Proposition 3.1.** If set $f_{ij} = \frac{\left(\begin{smallmatrix} a_{n+1,j} \\ a_{n+1,1} \end{smallmatrix}\right)}{a_{n+1,1}}$ for $i \neq j$, $f_{ii} = 0$ for all $1 \leq i \leq m$ and $F = (f_{ij})$, the Gauss-Codazzi-Ricci equations for the immersion $X$ are the $G_{m,n}^1$-II system (2.5) for $(A_1, F)$ which is called the generalized Laplace equation in [3].

Suppose

$$X = e_{m+n+1}, \quad e_k = \frac{X_{x_k}}{a_{n+1,k}}, \quad 1 \leq k \leq m$$

and

$$g = (e_1, \ldots, e_{m+n+1}) \in O(m+n, 1).$$

Then

$$g^{-1} dg = \Omega_\lambda|_{\lambda=1} = \sum_{i=1}^m \begin{pmatrix} C_i F^t - FC_i^t & -C_i A_1^t I_{n,1} \\ A_1 C_i^t & 0 \end{pmatrix} dx_i.$$

By the fundamental theorem of submanifolds, we have

**Proposition 3.2.** Let $(A_1, F)$ be a solution of the $G_{m,n}^1$-II system (2.5), then (3.2) is solvable. Let $g$ be a solution of (3.2) and $X$ the last column of $g$. If all entries of the last row of $A_1$ are non-zero, then $X$
is a local isometric immersion of a flat $m$-dimensional submanifold in hyperbolic space $\mathbb{H}^{m+n}$ with flat, non-degenerate normal bundle such that the two fundamental forms are given by (3.1), where $A_1 = (a_{ij})$.

**Theorem 3.3.** Let $E^{II}$ be a frame of the solution $(A_1, F)$ of the $G^{1}_{m,n}$-II system (2.5), $g_{s,\pi}$ given by (2.6) and $(\tilde{A}_1, \tilde{F}, E^{II}) = g_{s,\pi}'(A_1, F, E^{II})$.

Write
\[
E^{II}(x, 1) = (e_1, \ldots, e_m, e_{m+1}, \ldots, e_{m+n}, X),
\]
\[
\tilde{E}^{II}(x, 1) = (\tilde{e}_1, \ldots, \tilde{e}_m, \tilde{e}_{m+1}, \ldots, \tilde{e}_{m+n}, \tilde{X}).
\]

Then:

1. both $X$ and $\tilde{X}$ are flat $m$-dimensional submanifolds in $\mathbb{H}^{m+n}$ with flat and non-degenerate normal bundle, $\{x_1, \ldots, x_m\}$ line of curvature coordinates, $\{e_{m+k}\}_{k=1}^m$ and $\{\tilde{e}_{m+k}\}_{k=1}^m$ are parallel normal frames for $X$ and $\tilde{X}$ respectively.

2. The bundle morphism $P : \partial(X) \rightarrow \partial(\tilde{X})$ defined by $P(e_{m+k}(x)) = \tilde{e}_{m+k}(\partial(x))$ for $1 \leq k \leq n$ is a Ribaucour transformation covering the map $\mathcal{L} : X(x) \rightarrow \tilde{X}(x)$.

**Proof.** (1) follows from lemma 2.2 and proposition 3.2.

(2) Firstly we show that if $(A_1, F)$ is a solution of the $G^{1}_{m,n}$-II system (2.5), then there exists an $\mathcal{M}_{n \times (n+1-m)}$-valued map $G$ such that $\xi = (F, G)$ is a solution of the $G^{1}_{m,n}$-system (2.2). We consider two cases.

**Case 1.** $n = m - 1$. In this case $A_1 \in O(n, 1)$ and we only need to prove that $F$ is a solution of the $G^{1}_{m,n}$-system (2.2). Let $h_1 = \begin{pmatrix} I_m & 0 \\ 0 & A_1^{-1} \end{pmatrix}$ and the gauge transformation of $h_1$ on $\Omega_\lambda$ is

\[
h_1 * \Omega_\lambda = h_1 \Omega_\lambda h_1^{-1} - dh_1 h_1^{-1} = \sum_{i=1}^{m} \begin{pmatrix} C_i F_i - FC_i^t \\ C_i^t \end{pmatrix} dx_i + \begin{pmatrix} -C_i I_{n,1} \lambda \\ A_1^{-1} A_{1x_i} \end{pmatrix}
\]

By using (2.5), we have

\[
A_1^{-1} A_{1x_i} - (C_i F_i - I_{n,1} F_i C_i I_{n,1}) = Y C_i, \quad 1 \leq i \leq m,
\]

where $Y : R^m \rightarrow gl(m)$. It follows from (3.5) and $A_1 \in O(n, 1)$ that $Y C_i I_{n,1} + (Y C_i I_{n,1})^t = 0$ for all $1 \leq i \leq n$, which implies that $Y = 0$. Hence $F$ is a solution of the $G^{1}_{m,n}$-system (2.2).

**Case 2.** $n > m - 1$. Choose $A_2 \in \mathcal{M}_{(n+1) \times (n+1-m)}$ such that $A = (A_1, A_2) \in O(n, 1)$. Set $h = \begin{pmatrix} I_m & 0 \\ 0 & A^{-1} \end{pmatrix}$. The gauge transformation
of \( h \) on \( \Omega_\lambda \) is
\[
h \ast \Omega_\lambda = h\Omega_\lambda h^{-1} - dhh^{-1}
\]
(3.6) \[
= \sum_{i=1}^{m} \begin{pmatrix} C_i F^i - FC_i^i & -C_i \lambda \\ C_i \lambda & A^i_1 I_{n-1} A_{1i} \\ 0 & JA^i_2 I_{n-1} A_{1i} \end{pmatrix} dx_i,
\]
where \( J = I_{n-m,1} \). From (2.5), we have
\[
dA_1 = A_1 \sum_{i=1}^{m} (C_i F^i - F^i C_i) dx_i + \zeta \sum_{i=1}^{m} C_i dx_i
\]
for some \( \zeta : R^m \to M_{(n+1) \times m} \). Thus
\[
A^i_2 I_{n-1} dA_1 = A^i_2 I_{n-1} \zeta \sum_{i=1}^{m} C_i dx_i.
\]
Notice that \( A^{-1} dA \) is flat and
\[
A^{-1} dA = \begin{pmatrix} A^i_1 I_{n-1} A_{1i} & A^i_1 I_{n-1} A_{2i} \\ JA^i_2 I_{n-1} A_{1i} & JA^i_2 I_{n-1} A_{2i} \end{pmatrix},
\]
we can get
\[
JdA^i_2 \wedge I_{n-1} dA_2 = JA^i_2 I_{n-1} dA_1 \wedge A^i_2 I_{n-1} dA_2 + JA^i_2 I_{n-1} dA_2 \wedge JA^i_2 I_{n-1} dA_2.
\]
It follows from
\[
A^i_1 I_{n-1} dA_2 = (dA^i_2 I_{n-1} A_1)^t = -(A^i_2 I_{n-1} dA_1)^t = -\sum_{i=1}^{m} C_i \zeta^i J A^i_2 dx_i,
\]
that \( JA^i_2 I_{n-1} dA_2 \) is flat. Therefore there exists \( h_2 : R^m \to O(n - m, 1) \) such that \( h_2^{-1} dh_2 = JA^i_2 I_{n-1} dA_2 \). Take a gauge transformation by
\[
\tilde{h}_2 = \begin{pmatrix} I_m & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & h_2 \end{pmatrix}
\]
on \( h \ast \Omega_\lambda \), the resulting 1-form is
\[
\tilde{h}_2 \ast (h \ast \Omega_\lambda) = \tilde{h}_2 (h \ast \Omega_\lambda) \tilde{h}_2^{-1} - d\tilde{h}_2 \tilde{h}_2^{-1}
\]
\[
= \sum_{i=1}^{m} \begin{pmatrix} C_i F^i - FC_i^i & -C_i \lambda \\ C_i \lambda & A^i_1 I_{n-1} A_{1i} \\ 0 & JA^i_2 I_{n-1} A_{1i} \end{pmatrix} \zeta C_i \tilde{h}_2 \]
Set \( G = -\zeta^i I_{n-1} A_2 \tilde{h}_2^{-1} \) and using (3.7), we have
\[
A^i_1 I_{n-1} A_{1i} - (C_i^i F - I_{n-1} F^i C_i I_{n-1}) = Y C_i, \quad 1 \leq i \leq m,
\]
where \( Y = A^1_i I_{n,1} \zeta \). By using (3.10) and \( A^1_i I_{n,1} A_1 = I_m \), we get \( YC_i + (YC_i)' = 0 \) for all \( 1 \leq i \leq n \) which implies \( Y = 0 \). Hence \( \xi = (F, G) \) is a solution of the \( G_{m,n}^1 \)-system (2.2).

By using proposition 3.2, we know \((A_1, F)\) and \((\tilde{A}_1, \tilde{F})\) are solutions of the \( G_{m,n}^1 \)-II system (2.5) corresponding to \( X \) and \( \tilde{X} \) respectively. Let \( A_2, G, A = (A_1, A_2) \) be given as above and \( \tilde{W}, \tilde{Z} \) as in lemma 2.2. Let

\[
\gamma = (\gamma_1, \ldots, \gamma_{m+n+1}) = (\cos \rho \hat{W}', \sin \rho \hat{Z}' A')
\]

where \( \cos \rho = \frac{s}{\sqrt{1 + s^2}} \) and \( \sin \rho = \frac{-1}{\sqrt{1 + s^2}} \). Substituting \( \lambda = 1 \) into (2.8), we obtain

\[
\tilde{E}^{II}(x, 1) = E^{II}(x, 1)(I_{m+n+1} - 2\gamma \gamma I_{m+n,1}).
\]

Substituting (3.3) into (3.12), we have

\[
\hat{e}_k = e_k - 2\gamma_k \sum_{j=1}^{m+n+1} \gamma_j e_j, \quad k = 1, \ldots, m+n,
\]

\[
\tilde{X} = X + 2\gamma_{m+n+1} \sum_{j=1}^{m+n+1} \gamma_j e_j.
\]

Notice that \( X = e_{m+n+1} \) and \( \tilde{X} = \hat{e}_{m+n+1} \), we know

\[
\gamma_k X - \gamma_{m+n+1} e_k = \gamma_k \tilde{X} - \gamma_{m+n+1} \hat{e}_k, \quad k = 1, \ldots, m+n.
\]

Let \( \Gamma_k = \arctanh \frac{\gamma_{m+n+1}}{\gamma_k} \), \( 1 \leq k \leq n+m \), (3.14) becomes

\[
\cosh \Gamma_k \tilde{X} - \sinh \Gamma_k e_k = \cosh \Gamma_k \hat{X} - \sinh \Gamma_k \hat{e}_k.
\]

Geometrically (3.19) means that the geodesic of \( \mathbb{H}^{m+n} \) at \( X(x) \) in the direction \( e_k(x) \) intersects the geodesic of \( \mathbb{H}^{m+n} \) at \( \tilde{X} \) in the direction \( \hat{e}_k(x) \) at a point equidistant to \( X(x) \) and \( \tilde{X}(x) \). Thus the bundle morphism \( P : \vartheta(X) \to \vartheta(\tilde{X}) \) is a Ribaucour transformation.

**Acknowledgement.** The author would like to thank prof. Joonsang Park and anonymous referees for useful suggestions and comments. This work is partially supported by the Postdoctoral research fellowship at KIAS and the National Natural Science Foundation of China (10501043).
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