ON FUZZY BITOPOLITICAL SPACES IN ŠOSTAK’S SENSE

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Abstract. In this paper, we used the supra fuzzy topology which generated from a fuzzy bitopological space [1] to introduce and study the concepts of continuity (resp. openness, closeness) of mapping, separation axioms and compactness for a fuzzy bitopological spaces. Our definition preserve much of the correspondence between concepts of fuzzy bitopological spaces and the associated fuzzy topological spaces.

1. Introduction and preliminaries

Šostak [16], introduce the fundamental concept of fuzzy topological structure as an extension of both crisp topology and Chang’s fuzzy topology [5], in the sense that not only the object were fuzzified, but also the axiomatics. In [17, 18] Šostak gave some rules and showed how such an extension can be realized. Chattopdhyay et al. [6, 7] have redefined the similar concept. In [15, 8] Ramadan gave a similar definition namely “Smooth fuzzy topology” for lattice $L = [0, 1]$, it has been developed in many direction [4, 10-13, 17, 18]. Ghanim et al. [9] introduce the supra fuzzy topology as an extension of supra fuzzy topology in sense of Abd El-Monsef and Ramadan [2]. Abbas [1] generated the supra fuzzy topology from a fuzzy bitopological spaces. In this paper we have used the supra fuzzy topology which created from fuzzy bitopological spaces to introduce and study the concepts of continuity of mapping, separation axioms and compactness of the fuzzy bitopological spaces.

Throughout this paper, let $X$ be a nonempty set $I = [0, 1]$, $I_0 = (0, 1]$ and $I^X$ denote the set of all fuzzy subsets of $X$. $FP$ (resp. $FP^*$)
stand for fuzzy pairwise (resp. fuzzy $P^*$). For $\alpha \in I$, $\underline{\alpha}(x) = \alpha$ for all $x \in X$. A fuzzy point $x_t$ in $X$ is a fuzzy set taking value $t \in I_0$ at $x$ and zero elsewhere, $x_t \in \lambda$ if and only if $t \leq \lambda(x)$. A fuzzy set $\lambda$ is quasi-coincident with a fuzzy set $\mu$, denoted by $\lambda \sim_{\mu}$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. Otherwise $\lambda \not\sim_{\mu}$.

**Definition 1.1** [9, 16]. A mapping $\tau : I^X \rightarrow I$ is called supra fuzzy topology on $X$ if it satisfies the following conditions:

1. $(S1) \quad \tau(0) = \tau(1) = 1$.
2. $(S2) \quad \tau(\bigwedge_{i \in J} \mu_i) \geq \bigwedge_{i \in J} \tau(\mu_i)$, for any $\{\mu_i : i \in J\} \subseteq I^X$.

The pair $(X, \tau)$ is called supra fuzzy topological space (briefly, sfts). A supra fuzzy topology $\tau$ is called fuzzy topology on $X$ if

$(T) \quad \tau(\mu_1 \land \mu_2) \geq \tau(\mu_1) \land \tau(\mu_2)$, for any $\mu_1, \mu_2 \in I^X$, and the pair $(X, \tau)$ is called fuzzy topological space (briefly, fts). The triple $(X, \tau_1, \tau_2)$ is called fuzzy bitopological space (briefly, fbts) where, $\tau_1$ and $\tau_2$ are fuzzy topologies on $X$.

Throughout this paper, the indices $i, j \in \{1, 2\}$ and $i \neq j$.

**Definition 1.2** [11]. A mapping $f : (X, \tau_1, \tau_2) \rightarrow (Y, \tau^*_1, \tau^*_2)$ from a fbts $(X, \tau_1, \tau_2)$ to another fbts $(Y, \tau^*_1, \tau^*_2)$ is said to be:

(i) $FP$-continuous if and only if $\tau_i(f^{-1}(\mu)) \geq \tau^*_i(\mu)$ for each $\mu \in I^Y$ and $i = 1, 2$.
(ii) $FP$-open if and only if $\tau^*_i(f(\mu)) \geq \tau_i(\mu)$ for each $\mu \in I^X$ and $i = 1, 2$.
(iii) $FP$-closed if and only if $\tau^*_i(1 - f(\mu)) \geq \tau_i(1 - \mu)$ for each $\mu \in I^X$ and $i = 1, 2$.

**Theorem 1.1** [3]. Let $(X, T)$ be an ordinary topological space (resp. supra topological space). Then, the mapping $\omega(T) : I^X \rightarrow I$ defined by

$$\omega(T)(\lambda) = \bigvee \{\alpha \in I : \lambda^{-1}(\alpha, 1] \subseteq T\}$$

for every $\lambda \in I^X$ is fuzzy topology (resp. supra fuzzy topology) on $X$.

This provides a “goodness of extension” criterion for fuzzy topological properties. Recall that a fuzzy extension of a topological property of $(X, T)$ is said to be good when it is possessed by $\omega(T)$ if and only if the original property is possessed by $T$.

**Theorem 1.2** [3]. Let $(X, T)$ and $(Y, T^*)$ be ordinary topological spaces. If a mapping $f : (X, T) \rightarrow (Y, T^*)$ is continuous, then $f : (X, \omega(T)) \rightarrow (Y, \omega(T^*))$ is fuzzy continuous.
Definition 1.3 [1]. A mapping $C : I^X \times I_0 \rightarrow I^X$ is called supra fuzzy closure operator on $X$ if for $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following conditions:

(C1) $C(0, r) = 0$.
(C2) $\lambda \leq C(\lambda, r)$.
(C3) $C(\lambda, r) \vee C(\mu, r) \leq C(\lambda \vee \mu, r)$.
(C4) $C(\lambda, r) \leq C(\lambda, s)$ if $r \leq s$.
(C5) $C(C(\lambda, r), r) = C(\lambda, r)$.

The pair $(X, C)$ is called supra fuzzy closure space.

Theorem 1.3 [1]. Let $(X, \tau)$ be a sfts. For each $\lambda \in I^X$, $r \in I_0$ we define a mapping $\tau_{\tau} : I^X \times I_0 \rightarrow I^X$ as follows:

$$\tau_{\tau}(\lambda, r) = \wedge\{\mu : \mu \geq \lambda, \tau(1 - \mu) \geq r\}.$$ 

Then, $(X, \tau_{\tau})$ is supra fuzzy closure space. The mapping $I_{\tau} : I^X \times I_0 \rightarrow I^X$ which defined by: $I_{\tau}(\lambda, r) = \{\mu : \mu \leq \lambda, \tau(\mu) \geq r\}$ satisfies $I_{\tau}(1 - \lambda, r) = 1 - C_{\tau}(\lambda, r)$.

Theorem 1.4 [1]. Let $(X, \tau_1, \tau_2)$ be a fbts. For each $\lambda \in I^X$, $r \in I_0$ we define a mapping $C_{12} : I^X \times I_0 \rightarrow I^X$ as follows:

$$C_{12}(\lambda, r) = C_{\tau_1}(\lambda, r) \wedge C_{\tau_2}(\lambda, r).$$ 

Then, $(X, C_{12})$ is supra fuzzy closure space. The mapping $I_{12} : I^X \times I_0 \rightarrow I^X$ which defined by $I_{12}(\lambda, r) = I_{\tau_1}(\lambda, r) \vee I_{\tau_2}(\lambda, r)$ satisfies $I_{12}(1 - \lambda, r) = 1 - C_{12}(\lambda, r)$.

Theorem 1.5 [1]. Let $(X, \tau_1, \tau_2)$ be a fbts. Let $(X, C_{12})$ be a supra fuzzy closure space. Define the mapping $\tau_s : I^X \rightarrow I$ on $X$ by

$$\tau_s(\lambda) = \vee\{\tau_1(\lambda_1) \wedge \tau_1(\lambda_2) : \lambda = \lambda_1 \vee \lambda_2\},$$ 

where $\vee$ is taken over all families $\{\lambda_1, \lambda_2 : \lambda = \lambda_1 \vee \lambda_2\}$. Then,

(i) $\tau_s = C_{C_{12}}$ is the coarsest supra fuzzy topology on $X$ which is finer than $\tau_1$ and $\tau_2$.

(ii) $C_{12} = C_{\tau_s} = C_{C_{12}}$.

2. $FP^*$-continuity

We are now going to use the family $\tau_s$ which is generated by the two fuzzy topologies $\tau_1$ and $\tau_2$ to introduce another type of fuzzy continuity (resp. openness, closeness) of the fuzzy pairwise mappings.
Definition 2.1. Let \( f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*) \) be a mapping from a fbts \((X, \tau_1, \tau_2)\) to another fbts \((Y, \tau_1^*, \tau_2^*)\). \( f \) is called \( FP^* \)-continuous (resp. \( FP^* \)-open, \( FP^* \)-closed) if and only if \( f : (X, \tau_s) \to (Y, \tau_s^*) \) is \( FP \)-continuous (resp. \( FP \)-open, \( FP \)-closed).

Theorem 2.1. Every \( FP \)-continuous (resp. \( FP \)-open, \( FP \)-closed) mapping is \( FP^* \)-continuous (resp. \( FP^* \)-open, \( FP^* \)-closed).

Proof. Let \( f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*) \) be a \( FP \)-continuous mapping from a fbts \((X, \tau_1, \tau_2)\) to another fbts \((Y, \tau_1^*, \tau_2^*)\) and \((X, \tau_s), (Y, \tau_s^*)\) their associated sfts’s. Suppose that there exists \( \mu \in I^Y \) and \( r \in I_0 \) such that
\[
\tau_s(f^{-1}(\mu)) < r \leq \tau_s^*(\mu).
\]
There exists \( \mu_1, \mu_2 \in I^Y \) with \( \mu = \mu_1 \lor \mu_2 \) such that \( \tau_s^*(\mu) = \tau_1^*(\mu_1) \land \tau_2^*(\mu_2) \geq r \). Then, \( \tau_1^*(\mu_1) \geq r \) and \( \tau_2^*(\mu_2) \geq r \). From \( FP \)-continuity we have
\[
\tau_1(f^{-1}(\mu_1)) \geq r \quad \text{and} \quad \tau_2(f^{-1}(\mu_2)) \geq r.
\]
This implies that
\[
\tau_1(f^{-1}(\mu_1)) \land \tau_2(f^{-1}(\mu_2)) \geq r.
\]
Since \( f^{-1}(\mu) = f^{-1}(\mu_1) \lor f^{-1}(\mu_2) \), we have \( \tau_s(f^{-1}(\mu)) \geq r \). It is a contradiction. Hence,
\[
\tau_s(f^{-1}(\mu)) \geq \tau_s^*(\mu), \quad \forall \mu \in I^Y.
\]
Thus, \( f \) is \( FP^* \)-continuous. The other parts can be proved by the same manner. \( \square \)

Example 2.1. Let \( X = \{a, b, c\} \) and \( Y = \{x, y\} \). Define \( \lambda_1, \lambda_2 \in I^X \) and \( \mu_1, \mu_2 \in I^Y \) as follows:
\[
\begin{align*}
\lambda_1(a) &= 0.3 & \lambda_1(b) &= 0.3 & \lambda_1(c) &= 0.5 \\
\lambda_2(a) &= 0.5 & \lambda_2(b) &= 0.5 & \lambda_2(c) &= 0.3 \\
\mu_1(x) &= 0.5 & \mu_1(y) &= 0.3 \\
\mu_2(x) &= 0.3 & \mu_2(y) &= 0.5
\end{align*}
\]
We define fuzzy topologies $\tau_1, \tau_2 : I^X \to I$ and $\tau_1^*, \tau_2^* : I^Y \to I$ as follows:

\[
\tau_1(\lambda) = \begin{cases}
1, & \text{if } \lambda = 0.1 \\
0.5, & \text{if } \lambda = \lambda_1 \\
0, & \text{otherwise,}
\end{cases} \quad \tau_2(\lambda) = \begin{cases}
1, & \text{if } \lambda = 0.1 \\
0.3, & \text{if } \lambda = \lambda_2 \\
0, & \text{otherwise,}
\end{cases}
\]

\[
\tau_1^*(\mu) = \begin{cases}
1, & \text{if } \mu = 0.1 \\
0.3, & \text{if } \mu = \mu_1 \\
0, & \text{otherwise,}
\end{cases} \quad \tau_2^*(\mu) = \begin{cases}
1, & \text{if } \mu = 0.1 \\
0.2, & \text{if } \mu = \mu_2 \\
0, & \text{otherwise.}
\end{cases}
\]

From fbt's $(X, \tau_1, \tau_2)$ and $(Y, \tau_1^*, \tau_2^*)$ we can induce supra fuzzy topologies $\tau_s$ and $\tau_s^*$ as follows:

\[
\tau_s(\lambda) = \begin{cases}
1, & \text{if } \lambda = 0.1 \\
0.5, & \text{if } \lambda = \lambda_1 \\
0.3, & \text{if } \lambda = \lambda_2 \\
0.3, & \text{if } \lambda = \lambda_1 \lor \lambda_2 \\
0, & \text{otherwise,}
\end{cases}
\]

\[
\tau_s^*(\mu) = \begin{cases}
1, & \text{if } \mu = 0.1 \\
0.3, & \text{if } \mu = \mu_1 \\
0.2, & \text{if } \mu = \mu_2 \\
0.2, & \text{if } \mu = \mu_1 \lor \mu_2 \\
0, & \text{otherwise.}
\end{cases}
\]

Consider the mapping $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ defined by:

\[
f(a) = x, \quad f(b) = x, \quad f(c) = y.
\]

Then, $f$ is $FP^*$-continuous but not $FP$-continuous.

**Example 2.2.** In the above example we define $\tau_1^*$ and $\tau_2^*$ as follows:

\[
\tau_1^*(\mu) = \begin{cases}
1, & \text{if } \mu = 0.1 \\
0.6, & \text{if } \mu = \mu_1 \\
0, & \text{otherwise,}
\end{cases} \quad \tau_2^*(\mu) = \begin{cases}
1, & \text{if } \mu = 0.1 \\
0.5, & \text{if } \mu = \mu_2 \\
0, & \text{otherwise.}
\end{cases}
\]
From fbts \((Y, \tau_1^*, \tau_2^*)\) we can induce the supra fuzzy topology \(\tau_2^*\) as follows:

\[
\tau_2^*(\mu) = \begin{cases} 
1, & \text{if } \mu = 0, 1 \\
0.6, & \text{if } \mu = \mu_1 \\
0.5, & \text{if } \mu = \mu_2 \\
0.5, & \text{if } \mu = \mu_1 \vee \mu_2 \\
0, & \text{otherwise.}
\end{cases}
\]

Then, the mapping \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \tau_1^*, \tau_2^*)\) is FP*-open but not FP-open.

**Theorem 2.2.** Let \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \tau_1^*, \tau_2^*)\) be a mapping from a fbts \((X, \tau_1, \tau_2)\) to another fbts \((Y, \tau_1^*, \tau_2^*)\). Then, the following conditions are equivalent:

(i) \(f\) is FP*-continuous.

(ii) \(\tau_s(1 - f^{-1}(\mu)) \geq \tau_s^*(1 - \mu)\) for each \(\mu \in I^Y\).

(iii) \(f(C_{12}(\lambda, r)) \leq C_{12}(f(\lambda), r)\) for each \(\lambda \in I^X\) and \(r \in I_0\).

(iv) \(C_{12}(f^{-1}(\mu), r) \leq f^{-1}(C_{12}(\mu, r))\) for each \(\mu \in I^Y\) and \(r \in I_0\).

(v) \(f^{-1}(I_{12}(\mu, r)) \leq I_{12}(f^{-1}(\mu), r)\) for each \(\mu \in I^Y\) and \(r \in I_0\).

**Proof.** (i) \(\Rightarrow\) (ii) It is easily proved from Definition 2.1, and the fact \(f^{-1}(1 - \mu) = 1 - f^{-1}(\mu)\).

(ii) \(\Rightarrow\) (iii) For each \(\lambda \in I^X\) and \(r \in I_0\) we have,

\[
f^{-1}(C_{12}(f(\lambda), r)) = f^{-1}(C_{\tau_2^*}^*(f(\lambda), r)) \quad \text{(from Theorem 1.5.)}
\]

\[
= f^{-1}[\land\{\eta \in I^Y : \eta \geq f(\lambda), \tau_2^*(1 - \eta) \geq r\}]
\]

\[
= \land\{f^{-1}(\eta) \in I^X : f^{-1}(\eta) \geq \lambda, \tau_2^*(1 - \eta) \geq r\}
\]

\[
\geq \land\{f^{-1}(\eta) \in I^X : f^{-1}(\eta) \geq \lambda, \tau_s(1 - f^{-1}(\eta)) \geq r\}
\]

\[
= C_{\tau_s}(\lambda, r) = C_{12}(\lambda, r).
\]

Thus, \(f(C_{12}(\lambda, r)) \leq C_{12}(f(\lambda), r)\).

(iii) \(\Rightarrow\) (iv) For all \(\mu \in I^Y, r \in I_0\), put \(\lambda = f^{-1}(\mu)\). From (iii) we have,

\[
f(C_{12}(f^{-1}(\mu), r)) \leq C_{12}(f(f^{-1}(\mu)), r) \leq C_{12}(\mu, r).
\]

This implies that

\[
C_{12}(f^{-1}(\mu), r) \leq f^{-1}(f(C_{12}(f^{-1}(\mu), r))) \leq f^{-1}(C_{12}(\mu, r)).
\]

(iv) \(\Rightarrow\) (v) For all \(\mu \in I^Y, r \in I_0\) we have

\[
C_{12}(f^{-1}(1 - \mu), r) \leq f^{-1}(C_{12}(1 - \mu, r)).
\]
This implies that
\[
1 - f^{-1}(C_{12}(1 - \mu, r)) \leq 1 - C_{12}(f^{-1}(1 - \mu), r).
\]
Therefore
\[
f^{-1}(1 - C_{12}(1 - \mu, r)) \leq 1 - C_{12}(f^{-1}(1 - \mu), r).
\]
By Theorem 1.4, we have
\[
f^{-1}(I_{12}(\mu, r)) \leq I_{12}(1 - f^{-1}(1 - \mu), r) = I_{12}(f^{-1}(\mu), r).
\]
(v) \Rightarrow (i) Suppose that there exists \( \mu \in I^Y \) and \( r \in I_0 \) such that
\[
\tau_s(f^{-1}(\mu)) < r \leq \tau^*_{s}(\mu).
\]
Then, there exist \( \mu_1, \mu_2 \in I^Y \) such that
\[
\tau^*_s(\mu) = \tau^*_1(\mu_1) \lor \tau^*_2(\mu_2) \geq r \quad \text{and} \quad \mu = \mu_1 \lor \mu_2.
\]
This implies that \( \tau^*_1(\mu_1) \geq r \) and \( \tau^*_2(\mu_2) \geq r \). Then,
\[
I_{\tau^*_1}(\mu_1, r) = \mu_1 \quad \text{and} \quad I_{\tau^*_2}(\mu_2, r) = \mu_2.
\]
From Theorem 1.4, we have
\[
I_{12}(\mu, r) = I_{\tau^*_1}(\mu_1, r) \lor I_{\tau^*_2}(\mu_2, r) = \mu_1 \lor \mu_2 = \mu.
\]
By (v) we have
\[
f^{-1}(\mu) = f^{-1}(I_{12}(\mu, r)) \leq I_{12}(f^{-1}(\mu), r).
\]
Thus,
\[
f^{-1}(\mu) = I_{12}(f^{-1}(\mu), r) = 1 - C_{12}(1 - f^{-1}(\mu), r)
\]
\[
= 1 - C_{\tau_s}(1 - f^{-1}(\mu), r) = I_{\tau_s}(f^{-1}(\mu), r).
\]
This implies that
\[
\tau_s(f^{-1}(\mu)) \geq r.
\]
It is a contradiction. So, \( \tau_s(f^{-1}(\mu)) \geq \tau^*_s(\mu) \) for each \( \mu \in I^Y \). Hence,
\[
f: (X, \tau_1, \tau_2) \to (Y, \tau^*_1, \tau^*_2) \text{ is } FP^*-\text{continuous.}
\]
3. Separation axioms for fuzzy bitopological spaces

Definition 3.3. A fbts \((X, \tau_1, \tau_2)\) is called:

(i) \(FPR_0\) if and only if \(x_t \not\in C_{\tau_1}(y_m, r)\) implies that \(y_m \not\in C_{\tau_1}(x_t, r)\).

(ii) \(FPR_1\) if and only if \(x_t \not\in C_{\tau_1}(y_m, r)\) implies that there exist \(\lambda, \mu \in I^X\) with \(\tau_1(\lambda) \geq r, \tau_2(\mu) \geq r\) such that \(x_t \in \lambda, y_m \in \mu \) and \(\lambda \not\in \mu\).

(iii) \(FPR_2\) if and only if \(x_t \not\in C_{\tau_1}(\rho, r)\) implies that there exist \(\lambda, \mu \in I^X\) with \(\tau_1(\lambda) \geq r, \tau_2(\mu) \geq r\) such that \(x_t \in \lambda, \rho \leq \mu\) and \(\lambda \not\in \mu\).

(iv) \(FPR_3\) if and only if \(\eta = C_{\tau_1}(\eta, r) \not\in \rho = C_{\tau_1}(\rho, r)\) implies that there exist \(\lambda, \mu \in I^X\) with \(\tau_1(\lambda) \geq r, \tau_2(\mu) \geq r\) such that \(\eta \leq \lambda, \rho \leq \mu\) and \(\lambda \not\in \mu\).

(v) \(FPT_0\) if and only if \(x_t \not\in \lambda y_m\) implies that there exists \(\lambda \in I^X\) such that for \(i = 1\) or \(2\) \(\tau_i(\lambda) \geq r\) and \(x_t \in \lambda, y_m \not\in \lambda \) and \(x_t \not\in \lambda\).

(vi) \(FPT_1\) if and only if \(x_t \not\in \lambda y_m\) implies that there exists \(\lambda \in I^X\) such that for \(i = 1\) or \(2\) \(\tau_i(\lambda) \geq r\) and \(x_t \in \lambda \) and \(y_m \not\in \lambda\).

(vii) \(FPT_2\) if and only if \(x_t \not\in \lambda y_m\) implies that there exist \(\lambda, \mu \in I^X\) with \(\tau_1(\lambda) \geq r, \tau_2(\mu) \geq r\) such that \(x_t \in \lambda, y_m \in \mu \) and \(\lambda \not\in \mu\).

(viii) \(FPT_3\) if and only if \(x_t \not\in \lambda y_m\) implies that there exist \(\lambda, \mu \in I^X\) with \(\tau_1(\lambda) \geq r, \tau_2(\mu) \geq r\) such that \(x_t \in \lambda, y_m \in \mu\) and \(C_{\tau_1}(\lambda, r) \not\in C_{\tau_1}(\mu, r)\).

(ix) \(FPT_3\) if and only if it is \(FPR_2\) and \(FPT_1\).

(x) \(FPT_4\) if and only if it is \(FPR_3\) and \(FPT_1\).

Now, by making use of the supra fuzzy topology \(\tau_s\) generated by the two fuzzy topologies \(\tau_1\) and \(\tau_2\), we introduce and study weaker forms of the fuzzy pairwise separation axioms \(FPT_i\) \(i = 0, 1, 2, 2_1, 3, 4\) and \(FPR_i\) \(i = 0, 1, 2\).

Definition 3.1. A fbts \((X, \tau_1, \tau_2)\) is called:

(i) \(FP^*_R\) if and only if its associated sfts \((X, \tau_s)\) is \(FR_i\), \(i = 0, 1, 2\).

(ii) \(FP^*_T\) if and only if its associated sfts \((X, \tau_s)\) is \(FT_i\), \(i = 0, 1, 2, 2_1, 3, 4\).

Theorem 3.1. Let \((X, \tau_1, \tau_2)\) be a fbts. Then, we have

(i) \(FPR_i \Rightarrow FP^*_R, i = 0, 1, 2\).

(ii) \(FPT_i \Rightarrow FP^*_T, i = 0, 1, 2, 2_1, 3\).

(iii) \(FPT_i \Leftrightarrow FP^*_T, i = 0, 1\).

Proof. (i) Let \((X, \tau_1, \tau_2)\) be \(FPR_0\) and let \(x_t \not\in C_{\tau_1}(y_m, r)\) from Theorem 1.5, we have \(x_t \not\in C_{\tau_1}(y_m, r)\). Also, by Theorem 1.4, we have
$x_t \notin [C_{\tau_1}(y_m, r) \land C_{\tau_2}(y_m, r)]$. Then

$x_t \in \mathbb{1} - [C_{\tau_1}(y_m, r) \land C_{\tau_2}(y_m, r)] = \mathbb{1} - C_{\tau_1}(y_m, r) \lor \mathbb{1} - C_{\tau_2}(y_m, r)].$

This implies that $x_t \in \mathbb{1} - C_{\tau_1}(y_m, r)$ or $x_t \in \mathbb{1} - C_{\tau_2}(y_m, r)$, therefore $x_t \notin C_{\tau_1}(y_m, r)$ or $x_t \notin C_{\tau_2}(y_m, r)$. Since $(X, \tau_1, \tau_2)$ is a $FPR_0$, we have

$$y_m \notin C_{\tau_1}(x_t, r) \lor y_m \notin C_{\tau_2}(x_t, r).$$

This implies that

$$y_m \notin [C_{\tau_1}(x_t, r) \land C_{\tau_2}(x_t, r)] = C_{12}(x_t, r).$$

Then, $y_m \notin C_{\tau_3}(x_t, r)$, so $(X, \tau_1, \tau_2)$ is $FPT_1$. (For $i = 1, 2$ the proof is similar).

(ii) Let $(X, \tau_1, \tau_2)$ be a $FPPT_{21}^2$ and $x_t \notin y_m$. Then there exist $\lambda, \mu \in I^X$ such that $x_t \in \lambda$, $y_m \in \mu$, $\tau_1(\lambda) \geq r$, $\tau_2(\mu) \geq r$ and $C_{\tau_1}(\lambda, r) \notin C_{\tau_2}(\mu, r)$. Since $C_{\tau_1} \leq C_{\tau_2}$ for $i = 1, 2$ we have, $C_{\tau_3}(\lambda, r) \land \mu \notin C_{\tau_2}(\mu, r)$. Then $(X, \tau_1, \tau_2)$ is a $FPPT_{21}^2$. (For $i = 0, 1, 2, 3$ the proof is similar).

(iii) Necessity: follows from (ii). Sufficiency: Let $(X, \tau_1, \tau_2)$ be $FPPT_1$ and $x_t \notin y_m$. Then, there exists $\lambda \in I^X$ such that $x_t \in \lambda$, $\tau_1(\lambda) \geq r$ and $y_m \notin \lambda$. Since $\tau_1(\lambda) \geq r$ then there exist $\lambda_1, \lambda_2 \in I^X$, such that

$$\tau_1(\lambda) = \tau_1(\lambda_1) \land \tau_1(\lambda_2), \quad \lambda_1 = \lambda \lor \lambda_2.$$

Then, $\tau_1(\lambda_1) \geq r$ and $\tau_1(\lambda_2) \geq r$. $x_t \in \lambda$ implies that $x_t \in \lambda_1$ or $x_t \in \lambda_2$. Also, $y_m \notin \lambda$ implies that $y_m \notin \lambda_1$ and $y_m \notin \lambda_2$. Thus

$$(x_t \in \lambda_1, \tau_1(\lambda) \geq r, \text{ and } y_m \notin \lambda_1) \lor (x_t \in \lambda_2, \tau_2(\lambda) \geq r, \text{ and } y_m \notin \lambda_2).$$

Hence, $(X, \tau_1, \tau_2)$ is $FPT_1$. (For $i = 0$ the proof is similar). □

**Lemma 3.1.** Let $(X, \tau_1, \tau_2)$ be a fts. Then:

(i) If $(X, \tau_1)$ or $(X, \tau_2)$ is $FT_i$, then $(X, \tau_1, \tau_2)$ is $FPPT_i$ $i = 0, 1, 2, 2\frac{1}{2}, 3$.

(ii) If $(X, \tau_1)$ or $(X, \tau_2)$ is $FR_i$, then $(X, \tau_1, \tau_2)$ is $FPPT_i$ $i = 0, 1, 2$.

**Proof.** The proof is easy. □

**Example 3.1.** Let $X = \{x, y\}$. We define fuzzy topologies $\tau_1, \tau_2 : I^X \to I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 
1, & \text{if } \lambda \in \{0, 1\} \\
\frac{1}{3}, & \text{if } \lambda \in \{x_\alpha, y_\alpha\}, \alpha \in (0, 1) \\
\frac{1}{2}, & \text{if } \lambda \in \{x_\alpha \lor y_\alpha, x_\alpha \lor y_1, x_1 \lor y_\alpha\}, \alpha \in (0, 1) \\
0, & \text{otherwise}
\end{cases}$$
\[ \tau_2(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{0, 1\} \\ \frac{1}{2}, & \text{if } \lambda = \alpha, \alpha \in \left(0, \frac{1}{2}\right] \\ 0, & \text{otherwise} \end{cases} \]

Then, for \(0 < r \leq \frac{1}{3}\) the fts \((X, \tau_1, \tau_2)\) is \(FP^*T_3\) since, \((X, \tau_1)\) is \(FT_1\) and \(FR_2\) (see Lemma 3.1,) but \((X, \tau_1, \tau_2)\) is not \(FPT_3\).

**Example 3.2.** Let \(X = \{x, y, z\}\). We define fuzzy topologies \(\tau_1, \tau_2 : I^X \rightarrow I\) as follows:

\[ \tau_1(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{0, 1\} \\ \frac{1}{2}, & \text{if } \lambda = x, \alpha \in \left(0, \frac{1}{2}\right] \\ 0, & \text{otherwise} \end{cases} \]

\[ \tau_2(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{0, 1\} \\ \frac{1}{3}, & \text{if } \lambda = \{x, y, z\}, \alpha \in \{0, 1\} \\ \frac{1}{2}, & \text{if } \lambda = \{x \lor y, x \lor z, y \lor z\}, \alpha \in \{0, 1\} \\ 0, & \text{otherwise} \end{cases} \]

Then, for \(0 < r \leq \frac{1}{3}\) the fts \((X, \tau_2)\) is \(FT_2\) and from Lemma 3.1, we have \((X, \tau_1, \tau_2)\) is \(FP^*T_2\), but it is not \(FPT_2\).

**Example 3.3.** Let \(X = \{x, y\}\). We define fuzzy topologies \(\tau_1, \tau_2 : I^X \rightarrow I\) as follows:

\[ \tau_1(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{0, 1\} \\ \frac{1}{2}, & \text{if } \lambda = \{x_1, y_1\} \\ 0, & \text{otherwise} \end{cases} \]

\[ \tau_2(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{0, 1\} \\ \frac{2}{3}, & \text{if } \lambda = \{x_0.5, 0.5\} \\ 0, & \text{otherwise} \end{cases} \]

Then, for \(0 < r \leq \frac{1}{3}\) the fts \((X, \tau_1)\) is \(FT_{2\frac{1}{2}}\) and from Lemma 3.1, we have \((X, \tau_1, \tau_2)\) is \(FP^*T_{2\frac{1}{2}}\), but it is not \(FPT_{2\frac{1}{2}}\).

**Example 3.4.** Let \(X = \{x, y\}\). We define fuzzy topologies \(\tau_1, \tau_2 : I^X \rightarrow I\) as follows:

\[ \tau_1(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{0, 1\} \\ \frac{1}{2}, & \text{if } \lambda = \{0.3, 0.4\} \\ 0, & \text{otherwise} \end{cases} \]
Then, for $0 < r \leq \frac{1}{3}$ the fts $(X, \tau_1)$ is $FR_2$ and from Lemma 3.1, we have $(X, \tau_1, \tau_2)$ is $FP^*R_2$, but it is not $FPR_2$.

**Example 3.5.** Let $X = \{x, y\}$. We define fuzzy topologies $\tau_1, \tau_2 : I^X \rightarrow I$ as follows:

\[
\tau_1(\lambda) = \begin{cases} 
1, & \text{if } \lambda \in \{0, 1\} \\
\frac{1}{4}, & \text{if } \lambda \in \{x_1, y_1\} \\
0, & \text{otherwise}
\end{cases} \quad \tau_2(\lambda) = \begin{cases} 
1, & \text{if } \lambda \in \{0, 1\} \\
\frac{2}{7}, & \text{if } \lambda = 0.3 \\
0, & \text{otherwise}.
\end{cases}
\]

Then, for $0 < r \leq \frac{1}{3}$ the fts $(X, \tau_1)$ is $FR_1$ (resp.$FR_0$) and from Lemma 3.1, we have $(X, \tau_1, \tau_2)$ is $FP^*R_1$, (resp. $FP^*R_0$) but it is not $FPR_1$, (resp. $FPR_0$).

**Lemma 3.2.** Let $(X, \tau_1, \tau_2)$ be a fts. For $r \in I_0$ we have the following:

(i) For all $\lambda \in I^X$ with $\tau_1(\lambda) \geq r$, $\lambda q \mu$ if and only if $\lambda q C_{12}(\mu, r)$, $\mu \in I^X$.

(ii) $x_i q C_{12}(\lambda, r)$ if and only if $\lambda q \mu$ for all $\mu \in I^X$ with $\tau_2(\mu) \geq r$ and $x_i \in \mu$.

**Proof.** (i) Let $\lambda \in I^X$ with $\tau_1(\lambda) \geq r$, $\lambda q \mu$. Since $\mu \leq C_{12}(\mu, r)$, $\lambda q C_{12}(\mu, r)$. Conversely, let $\lambda \in I^X$ with $\tau_1(\lambda) \geq r$ and suppose that $\lambda \not\in q \mu$, then $\mu \leq 1 - \lambda$ this implies that

\[
C_{\tau_1}(\mu, r) \leq C_{\tau_2}(1 - \lambda, r) = 1 - \lambda.
\]

By using Theorem 1.5, we have $C_{12}(\mu, r) \leq 1 - \lambda$. Then, $\lambda \not\in q C_{12}(\mu, r)$. This is a contradiction.

(ii) Let $x_i q C_{12}(\lambda, r)$. Since $x_i \in \mu$, $\mu q C_{12}(\lambda, r)$ . By (i) we have $\mu q \lambda$ for all $\mu \in I^X$ with $\tau_2(\mu) \geq r$ and $x_i \in \mu$. Conversely, suppose that $x_i \not\in q C_{12}(\lambda, r)$. Then, $x_i \in 1 - C_{12}(\lambda, r)$. Let $\mu = 1 - C_{12}(\lambda, r)$. By Theorem 1.5, $\mu = 1 - C_{\tau_1}(\lambda, r)$, then $\tau_2(\mu) \geq r$. Since, $\lambda \leq C_{12}(\lambda, r)$, then $\mu = 1 - C_{12}(\lambda, r) \leq 1 - \lambda$ this implies that $\lambda \not\in q \mu$ a contradiction.\(\square\)

**Theorem 3.2.** Let $(X, \tau_1, \tau_2)$ be a fts. Then the following statements are equivalent:

(i) $(X, \tau_1, \tau_2)$ is $FP^*R_0$. 
(ii) $C_{12}(x_t, r) \leq \mu$ for all $\mu \in I^X, r \in I_0$ with $\tau_s(\mu) \geq r$ and $x_t \in \mu$

(iii) If $x_t \not\in \lambda = C_{\tau_s}(\lambda, r)$, there exits $\mu \in I^X$ with $\tau_s(\mu) \geq r$ such that $x_t \not\in \mu$ and $\lambda \leq \mu, r \in I_0$.

(iv) If $x_t \not\in \lambda = C_{\tau_s}(\lambda, r)$ then, $C_{\tau_s}(x_t, r) \not\in \lambda = C_{\tau_s}(\lambda, r), \lambda \in I^X, r \in I_0$.

(v) If $x_t \not\in C_{\tau_s}(y_m, r)$ then, $C_{\tau_s}(x_t, r) \not\in C_{\tau_s}(y_m, r), r \in I_0$.

**Proof.** (i) $\Rightarrow$ (ii) Let $y_m q C_{12}(x_t, r)$. By Theorem 1.5, we have $y_m q C_{\tau_s}(x_t, r)$. By using (i) we obtain $x_t q C_{\tau_s}(y_m, r)$ i.e. $x_t q C_{12}(y_m, r)$. By Lemma 3.2(ii), we can found that, $y_m q \mu$ for all $\mu \in I^X$ with $\tau_s(\mu) \geq r$ and $x_t \in \mu$. Then, we have $C_{12}(x_t, r) \leq \mu$ for all $\mu \in I^X, r \in I_0$ with $\tau_s(\mu) \geq r$ and $x_t \in \mu$.

(ii) $\Rightarrow$ (i) If $y_m \not\in C_{\tau_s}(x_t, r)$, we have $y_m \in 1 - C_{\tau_s}(x_t, r)$. By (ii) and the fact $\tau_s(1 - C_{\tau_s}(x_t, r)) \geq r$ we obtain

$$C_{12}(y_m, r) \leq 1 - C_{\tau_s}(x_t, r) \leq 1 - x_t.$$

Thus

$$x_t \not\in C_{12}(y_m, r) = C_{\tau_s}(y_m, r).$$

Hence, $(X, \tau_1, \tau_2)$ is $FP^* R_0$.

(i) $\Rightarrow$ (iii) Let $x_t \not\in \lambda = C_{\tau_s}(\lambda, r)$. Since $C_{\tau_s}(y_m, r) \leq C_{\tau_s}(\lambda, r)$ for each $y_m \in \lambda$ we have $x_t \not\in C_{\tau_s}(y_m, r)$. By (i) we have $y_m \not\in C_{\tau_s}(x_t, r)$. By Lemma 3.2(ii), for each $y_m \not\in C_{\tau_s}(x_t, r)$, there exists $\eta_{y_m} \in I^X$ such that $x_t \not\in \eta_{y_m}, \tau_s(\eta_{y_m}) \geq r, y_m \in \eta_{y_m}$. Let

$$\mu = \bigvee_{y_m \in \lambda} \{ \eta_{y_m} : x_t \not\in \eta_{y_m} \}.$$

From Definition 1.1, we have $\tau_s(\mu) \geq r$. Then,

$$x_t \not\in \mu, \lambda \leq \mu, \tau_s(\mu) \geq r.$$

(iii) $\Rightarrow$ (iv) Let $x_t \not\in \lambda = C_{\tau_s}(\lambda, r)$. By (iii) there exists $\mu \in I^X$ such that $x_t \not\in \mu, \lambda \leq \mu, \tau_s(\mu) \geq r$.

Since $x_t \not\in \mu$, it follows that $x_t \in 1 - \mu$, this implies that

$$C_{\tau_s}(x_t, r) \leq C_{\tau_s}(1 - \mu, r) = 1 - \mu \leq 1 - \lambda.$$

Hence, $C_{\tau_s}(x_t, r) \not\in \lambda = C_{\tau_s}(\lambda, r)$. 

(iv) $\Rightarrow$ (v) Let $x_t \not\in C_{\tau_s}(y_m, r)$. Since, $C_{\tau_s}(C_{\tau_s}(y_m, r), r) = C_{\tau_s}(y_m, r)$ and by using (iv) we have $C_{\tau_s}(x_t, r) \not\in C_{\tau_s}(y_m, r)$.

(v) $\Rightarrow$ (i) Let $x_t \not\in C_{\tau_s}(y_m, r)$. By (v) we have $C_{\tau_s}(x_t, r) \not\in C_{\tau_s}(y_m, r)$ and since $y_m \leq C_{\tau_s}(y_m, r)$ then $y_m \not\in C_{\tau_s}(x_t, r)$. Hence, $(X, \tau_1, \tau_2)$ is $FP^*R_0$. □

Lemma 3.3. Let $(X, T_1, T_2)$ be an ordinary topological space. Let $\omega(T_1)$ and $\omega(T_2)$ be the induced fuzzy topologies of $T_1$ and $T_2$ respectively. Also, let $\omega(T_s)$ be the induced supra fuzzy topology of the supra topology $T_s$ and let

$$(\omega(T))_s(\lambda) = \lor\{\omega(T_1)(\lambda_1) \land \omega(T_2)(\lambda_2) : \lambda = \lambda_1 \lor \lambda_2\}.$$ 

Then,

$$\omega(T_s) \supseteq (\omega(T))_s$$

Proof. Suppose that there exists $\lambda \in I^X$ and $r_0 \in I_0$ such that

$$(\omega(T))_s(\lambda) \geq r_0 > \omega(T_s)(\lambda).$$

Then, there exist $\lambda_1, \lambda_2 \in I^X$ such that $\lambda = \lambda_1 \lor \lambda_2$ with

$$\omega(T_1)(\lambda_1) \geq r_0 \quad \text{and} \quad \omega(T_2)(\lambda_2) \geq r_0.$$ 

Then

$$\lambda_1^{-1}(r_0, 1] \in T_1 \quad \text{and} \quad \lambda_2^{-1}(r_0, 1] \in T_2.$$ 

This implies that

$$\lambda^{-1}(r_0, 1] = \lambda_1^{-1}(r_0, 1] \cup \lambda_2^{-1}(r_0, 1] \in T_s.$$ 

Then, $(\omega(T_s))(\lambda) \geq r_0$ which is a contradiction. Hence,

$$(\omega(T))_s \subseteq \omega(T_s).$$ 

□

Theorem 3.3. Let $(X, T_1, T_2)$ be an ordinary bitopological space. 

(i) If $(X, \omega(T_1), \omega(T_2))$ is $FP^*T_i$, then $(X, T_1, T_2)$ is $P^*T_i$, $i = 0, 1, 2, 2^{1/2}, 3, 4$.

(ii) If $(X, \omega(T_1), \omega(T_2))$ is $FP^*R_i$, then $(X, T_1, T_2)$ is $P^*R_i$, $i = 0, 1, 2$. 
Proof. (i) (For \(i = 2\)): Let \(x \neq y\) and suppose that \(x_t \not\equiv y_m\). Let \(r \in I_0\) such that \(r < t, m\). Since \((X, \omega(T_1), \omega(T_2))\) is \(FP^*T_2\) there exist \(\lambda, \mu \in I^X\) with \((\omega(T))_s(\lambda) \geq r, (\omega(T))_s(\mu) \geq r\) such that \(x_t \in \lambda, y_m \in \mu\) and \(\lambda \not\equiv \mu\). By Lemma 3.3, we have

\[
\omega(T_s)(\lambda) \geq (\omega(T))_s(\lambda) \geq r.
\]

Then, \(\lambda^{-1}(r, 1) \in T_s\). Since \(x_t \in \lambda\) then, \(\lambda(x) \geq t > r\) implies that \(x \in \lambda^{-1}(r, 1)\). Similarly \(\mu^{-1}(r, 1) \in T_s\) and \(y \in \mu^{-1}(r, 1)\). Since, \(\lambda \not\equiv \mu\) then, \(\lambda \leq 1 - \mu\), implies that

\[
\lambda^{-1}(r, 1) \subseteq X - \mu^{-1}(r, 1).
\]

Then, \(\lambda^{-1}(r, 1) \cap \mu^{-1}(r, 1) = \phi\).

Hence, \((X, T_1, T_2)\) is \(P^*T_2\). (for, \(i = 0, 1, 2, 1, 2, 3, 4\) the proof is similar).

(ii) The proof is similar to part (i). \(\square\)

4. \(FP^*\)-compactness

Definition 4.1. Let \((X, \tau)\) be a fts and \(\mu \in I^X, r \in I_0\). Then :

(i) The family \(\{\eta_j : \tau(\eta_j) \geq r, j \in J\}\) is called \(\tau\)-cover of \(\mu\) if and only if for each \(x_t \in \mu\) there exists \(j_0 \in J\) such that \(x_t \in \eta_{j_0}\).

(ii) \(\mu\) is \(C\)-set if and only if every \(\tau\)-cover of \(\mu\) have a finite subcover.

(iii) \((X, \tau)\) is called \(F\)-compact if and only if for every \(\lambda \in I^X\) such that \(\tau(1 - \lambda) \geq r\) is \(C\)-set.

Definition 4.2. A fts \((X, \tau_1, \tau_2)\) is called \(FP^*\)-compact if and only if its associated sfts \((X, \tau_s)\) is \(F\)-compact.

Theorem 4.1. Let \((X, \tau_1, \tau_2)\) be a fts. If \((X, \tau_1)\) or \((X, \tau_2)\) is \(F\)-compact, then \((X, \tau_1, \tau_2)\) is \(FP^*\)-compact.

Proof. Let \((X, \tau_1)\) be \(F\)-compact. Let \(\lambda \in I^X\) such that \(\tau_s(1 - \lambda) \geq r, r \in I_0\) and \(\{\eta_j : \tau_s(\eta_j) \geq r, j \in J\}\) be a \(\tau_s\)-cover of \(\lambda\). Since \(\tau_s(1 - \lambda) \geq r\), then we can write

\[
\lambda = \lambda_1 \land \lambda_2, \quad \tau_i(1 - \lambda_i) \geq r, \quad (i = 1, 2).
\]

Then, for every \(x_t \in \lambda\), there exists \(\eta_{j_0} \in I^X\) with \(\tau_s(\eta_{j_0}) \geq r\) such that

\[
x_t \in \eta_{j_0} = \eta^{(1)}_{j_0} \lor \eta^{(2)}_{j_0}.
\]
for some \( n_i \in X \) with \( \tau_1(n_i) \geq r, \) \((i = 1, 2)\), then \( x_i \in \eta_j(i) \) or \( x_t \in \eta_j(j) \). Now, \( \{ \eta_j(i) : \tau_1(\eta_j(i)) \geq r, i = 1, 2, 3, \ldots \} \) is a \( \tau_1 \)-cover of \( \lambda_1 \) or \( \{ \eta_j(j) : \tau_2(\eta_j(j)) \geq r, i = 1, 2, 3, \ldots \} \) is a \( \tau_2 \)-cover of \( \lambda_2 \). If \((X, \tau_1)\) is \( F \)-compact, then \( \lambda_1 \) is \( C \)-set. So, there exists a finite \( \tau_1 \)-cover \( \{ \eta_i(i) : i = 1, 2, 3, \ldots, n \} \) of \( \lambda_1 \), this implies that

\[
\lambda \leq \lambda_1 \leq \bigvee_{i=1}^{n} \eta_i(i).
\]

Hence, \( \lambda \) is \( C \)-set consequently, the ftbs \((X, \tau_1, \tau_2)\) is \( FP^* \)-compact. Similarly, if \((X, \tau_2)\) is \( F \)-compact then, \((X, \tau_1, \tau_2)\) is \( FP^* \)-compact. \( \square \)

**Theorem 4.2.** Let \((X, \tau_1, \tau_2)\) be a \( FP^*T_3 \) and \( \mu \in X \) is \( C \)-set. Then, for every \( \lambda \in X \) with \( \tau_s(1-\lambda) \geq r, r \in I_0 \) such that \( \mu \neq \mu \), there is \( \nu, \rho \in X \) such that \( \tau_s(\nu) \geq r, \tau_s(\rho) \geq r \) and \( \eta \neq \rho \).

**Proof.** Since \( \nu \neq \mu \), then for each \( x_t \in \mu \) we have, \( x_t / \nu \lambda = C_\tau(\lambda, r), r \in I_0 \). Since \((X, \tau_1, \tau_2)\) is \( FP^*T_3 \), there exists \( \eta, \rho \in X \) with \( \tau_s(\eta) \geq r, \tau_s(\rho) \geq r \) such that \( x_t \in \eta \), \( \lambda \leq \rho \) and \( \eta \neq \rho \). Then, \( \{ \eta : \tau_s(\eta) \geq r, x_t \in \mu \} \) is \( \tau_s \)-cover of \( \mu \). Since, \( \mu \) is a \( C \)-set, then \( \mu \leq \bigvee_{i=1}^{n} \eta(i) \). Let \( \eta = \bigvee_{i=1}^{n} \eta(i) \) and \( \rho = \rho(x_t(i)) \) for all \( i \). Then \( \eta \neq \rho \). \( \square \)

**Theorem 4.3.** Let \((X, \tau_1, \tau_2)\) be a \( FP^*T_2 \), \( x_t \) be any fuzzy point of \( X \) and \( \lambda \in X \) is a \( C \)-set such that \( x_t \neq \lambda \). Then, there exist \( \eta_1, \eta_2 \in X \) with \( \tau_s(\eta_1) \geq r, \tau_s(\eta_2) \geq r, r \in I_0 \) such that \( x_t \in \eta_1, \lambda \leq \eta_2 \) and \( \eta_1 \neq \eta_2 \). Moreover, if \( \lambda, \mu \in X \) are \( C \)-sets such that \( \lambda \neq \mu \), then there exist \( \rho_1, \rho_2 \in X \) with \( \tau_s(\rho_1) \geq r, \tau_s(\rho_2) \geq r, r \in I_0 \) such that \( \lambda \leq \rho_1, \mu \leq \rho_2 \) and \( \rho_1 \neq \rho_2 \).

**Proof.** Since \( x_t \neq \lambda \), then \( x_t \neq \eta y \) for each \( y \in \lambda \). Since \((X, \tau_1, \tau_2)\) is \( FP^*T_2 \), there exist \( \eta_1, \eta \in X \) with \( \tau_s(\eta_1) \geq r, \tau_s(\eta) \geq r, r \in I_0 \) such that \( x_t \in \eta_1, y \in \eta \) and \( \eta \neq \eta \). Then, \( \{ \eta : \tau_s(\eta) \geq r, y \in \lambda \} \) is a \( \tau_s \)-cover of \( \lambda \). Since \( \lambda \) is \( C \)-set, there exists a finite subcover \( \{ \eta(i) : i = 1, 2, 3, \ldots, n \} \) of \( \lambda \). Let \( \eta_2 = \bigvee_{i=1}^{n} \eta(i) \). Then

\[
\tau_s(\eta_2) = \tau_s(\bigvee_{i=1}^{n} \eta(i)) \geq \bigwedge_{i=1}^{n} \tau_s(\eta(i)) \geq r.
\]

Since \( \eta \neq \eta(i), (i = 1, 2, 3, \ldots, n), \) then \( \eta_1 \leq 1 - \eta(i), (i = 1, 2, 3, \ldots, n) \), this implies that

\[
\eta_1 \leq \bigwedge_{i=1}^{n} (1 - \eta(i)) = 1 - \bigvee_{i=1}^{n} \eta(i) = 1 - \eta_2.
\]

Then, \( \eta_1 \neq \eta_2 \). For the second part, let \( x_t \in \mu \), since \( \lambda \neq \mu \), then \( x_t \neq \lambda \), by the first part there exist \( \rho, \rho_2 \in X \) with \( \tau_s(\rho) \geq r, \tau_s(\rho_2) \geq r, r \in I_0 \).
such that \( x_t \in \rho^* \), \( \lambda \leq \rho_2 \) and \( \rho^* \not\parallel \rho_2 \). Then, \( \{ \rho^*: \tau_s(\rho^*) \geq r, x_t \in \mu \} \) is a \( \tau_s \)-cover of \( \mu \), so there exist a finite subcover \( \{ \rho^*(i): i = 1, 2, 3, \ldots, n \} \). Let \( \rho_1 = \vee_{i=1}^n \rho^*(i) \). Then
\[
\tau_s(\rho_1) = \tau_s(\vee_{i=1}^n \rho^*(i)) \geq \wedge_{i=1}^n \tau_s(\rho^*(i)) \geq r.
\]
Since \( \rho_2 \not\parallel \rho^*(i), (i = 1, 2, 3, \ldots, n) \), then \( \rho_2 \not\parallel \rho_1 \). \( \square \)

**Theorem 4.4.** Let \((X, \tau_1, \tau_2)\) be a \( FP^*T_2 \). If \( \eta \in I^X \) is a \( C \)-set, then \( C_{\tau_s}(\eta, r) = \eta, r \in I_0 \).

**Proof.** Let \( x_t \in 1 - \eta \). Then, \( x_t \not\parallel \eta \). Since \( \eta \) is \( C \)-set, then by Theorem 4.3, there exist \( \mu_{x_t}, \lambda \in I^X \) with \( \tau_s(\mu_{x_t}) \geq r, \tau_s(\lambda) \geq r, r \in I_0 \) such that \( x_t \in \mu_{x_t}, \eta \leq \lambda \) and \( \mu_{x_t} \not\parallel \lambda \). This implies that
\[
x_t \in \mu_{x_t} \leq 1 - \lambda \leq 1 - \eta.
\]
Thus
\[
1 - \eta = \vee \{ \mu_{x_t}: x_t \in 1 - \eta \}.
\]
Then, \( \tau_s(1 - \eta) \geq r \). Hence \( C_{\tau_s}(\eta, r) = \eta \).

**Theorem 4.5.** Let \((X, \tau_1, \tau_2)\) be a \( FP^* \)-compact and \( FP^*T_2 \). Then, it is \( FP^*T_4 \).

**Proof.** Since \((X, \tau_1, \tau_2)\) is \( FP^*T_2 \) it is clear that it is \( FP^*T_1 \). Remains we prove that \((X, \tau_1, \tau_2)\) is \( FP^*R_3 \), so let \( \lambda_1 = C_{\tau_s}(\lambda_1, r) \not\parallel \lambda_2 = C_{\tau_s}(\lambda_2, r) \). Then, \( \tau_s(1 - \lambda_1) \geq r, \tau_s(1 - \lambda_2) \geq r \) and since \((X, \tau_1, \tau_2)\) is \( FP^* \)-compact, then \( \lambda_1 \) and \( \lambda_2 \) are \( C \)-sets. Since, \( \lambda_1 \not\parallel \lambda_2 \), then by Theorem 4.3, there exist \( \rho_1, \rho_2 \in I^X \) with \( \tau_s(\rho_1) \geq r, \tau_s(\rho_2) \geq r, r \in I_0 \) such that \( \lambda_1 \leq \rho_1, \lambda_2 \leq \rho_2 \) and \( \rho_1 \not\parallel \rho_2 \). Thus, \((X, \tau_1, \tau_2)\) is \( FP^*R_3 \). Hence \((X, \tau_1, \tau_2)\) is \( FP^*T_4 \). \( \square \)

**Theorem 4.6.** Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \tau_1^*, \tau_2^*) \) be a \( FP^* \)-continuous mapping from a \( fbts \) \((X, \tau_1, \tau_2)\) to another \( fbts \) \((X, \tau_1^*, \tau_2^*)\). If \( \mu \in I^X \) is \( C \)-set, then \( f(\mu) \) is \( C \)-set in \( Y \).

**Proof.** Let \( \{ \eta_i : i \in J \} \) be a \( \tau_s^* \)-cover of \( f(\mu) \). Then \( f(\mu) \leq \vee_{i \in J} \eta_i, \tau_s^*(\eta_i) \geq r, \) for all \( i \), this implies that
\[
\mu \leq f^{-1}(f(\mu)) \leq f^{-1}(\vee_{i \in J} \eta_i) = \vee_{i \in J} f^{-1}(\eta_i).
\]
Since \( f \) is \( FP^* \)-continuous, then
\[
\tau_s(f^{-1}(\eta_i)) \geq \tau_s^*(\eta_i) \geq r.
\]
Then, \( \{ f^{-1}(\eta_i) : i \in J \} \) is a \( \tau_s \)-cover of \( \mu \) and since \( \mu \) is \( C \)-set, then \( \mu \leq \bigvee_{i=1}^{n} f^{-1}(\eta_i) \) this implies that

\[
 f(\mu) \leq f(\bigvee_{i=1}^{n} f^{-1}(\eta_i)) = \bigvee_{i=1}^{n} f(f^{-1}(\eta_i)) \leq \bigvee_{i=1}^{n} \eta_i.
\]

Hence, \( f(\mu) \) is \( C \)-set in \( Y \). \( \square \)

**Corollary 4.1.** The \( FP^* \)-continuous image of an \( FP^* \)-compact is \( FP^* \)-compact.

**References**


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