ANALYTIC FOURIER-FEYNMAN TRANSFORM AND FIRST VARIATION ON ABSTRACT WIENER SPACE

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Abstract. In this paper we express analytic Feynman integral of the first variation of a functional $F$ in terms of analytic Feynman integral of the product of $F$ with a linear factor and obtain an integration by parts formula for the analytic Feynman integral of functionals on abstract Wiener space. We find the Fourier-Feynman transform for the product of functionals in the Fresnel class $\mathcal{F}(B)$ with $n$ linear factors.

1. Introduction and preliminaries

The concept of an $L_1$ analytic Fourier-Feynman transform for functionals on classical Wiener space was introduced by Brue in [2]. In [4] Cameron and Storvick introduced an $L_2$ analytic Fourier-Feynman transform on classical Wiener space. In [13] Johnson and Skoug developed an $L_p$ analytic Fourier-Feynman transform theory for $1 \leq p \leq 2$ that extended the results in [2, 4] and gave various relationships between the $L_1$ and $L_2$ theories. In [10, 11, 12], Huffman, Park and Skoug defined a convolution product for functionals on classical Wiener space and they showed that the analytic Fourier-Feynman transform of convolution product is the product of transforms. In [3] Cameron obtained Wiener integral of first variation of functional $F$ in terms of the Wiener integral of the product with a linear factor. In [6] Cameron and Storvick applied the result to Feynman integral and then gave formulas for Feynman integral of functionals on classical Wiener space that belong to the Banach algebra $S'$ introduce by Cameron and Storvick in [5]. In [17]
Park, Skoug and Storvick found the Fourier-Feynman transform of functional $F$ from the Banach algebra $S$ after it has been multiplied with $n$ linear factors. Recently, Chang, Kim and Yoo established the relationships among Fourier-Feynman transform, first variation and convolution product on abstract Wiener space [8, 9]. In this paper we express analytic Feynman integral of the first variation of a functional $F$ in terms of analytic Feynman integral of the product of $F$ with a linear factor and obtain an integration by parts formula for the analytic Feynman integral of functionals on abstract Wiener space. We find the Fourier-Feynman transform for the product of functionals in the Fresnel class $F(B)$ with $n$ linear factors.

Let $(H, B, \nu)$ be an abstract Wiener space and let $\{e_j\}$ be a complete orthonormal system in $H$ such that the $e_j$’s are in $B^*$, the dual of $B$. For each $h \in H$ and $x \in B$, we define a stochastic inner product $(h, x)_{\sim}$ as follows;

$$ (h, x)_{\sim} = \begin{cases} \lim_{n \to \infty} \sum_{j=1}^{n} \langle h, e_j \rangle \langle x, e_j \rangle, & \text{if the limit exists} \\ 0, & \text{otherwise} \end{cases} $$

where $(\cdot, \cdot)$ is a natural dual pairing between $B$ and $B^*$. It is well known [14, 15] that for each $h(\neq 0)$ in $H$, $(h, \cdot)_{\sim}$ is a Gaussian random variable on $B$ with mean zero and variance $|h|^2$, that is,

$$ \int_B \exp \{i(h, x)_{\sim}\} d\nu(x) = \exp \{-\frac{1}{2} |h|^2\}. $$

Let $M(H)$ denote the space of complex-valued countably additive Borel measures on $H$. Under the total variation norm $\|\cdot\|$ and with convolution as multiplication, $M(H)$ is a commutative Banach algebra with identity [1].

A subset $E$ of $B$ is said to be scale-invariant measurable provided $\alpha E$ is measurable for each $\alpha > 0$, and a scale-invariant measurable set $N$ is said to be scale-invariant null provided $\nu(\alpha N) = 0$ for each $\alpha > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). If two functionals $F$ and $G$ are equal s-a.e., then we write $F \approx G$. For more detail, see [7]. For a functional $F$ on $B$, we denote by $[F]$ the equivalence class of functionals $G$ which are equal to $F$ s-a.e., that is,

$$ [F] = \{ G : G \approx F \}. $$

We now introduce the Fresnel class $F(B)$ of functionals on $B$. The space $F(B)$ is defined as the space of all equivalence classes of stochastic
Fourier transforms of elements of $M(H)$, that is,

$$\mathcal{F}(B) = \{ [F] : F(x) = \int_{H} \exp \{ i(h, x)^\sim \} d\sigma(h), x \in B, \sigma \in M(H) \}. \tag{1.3}$$

As is customary, we will identify a function with its $s$-equivalence class and think of $\mathcal{F}(B)$ as a collection of functionals on $B$ rather than as a collection of equivalence classes.

It is well-known [14, 15] that $\mathcal{F}(B)$ is a Banach algebra with the norm $\|F\| = \|\sigma\|$ and the mapping $\sigma \mapsto F$ is a Banach algebra isomorphism where $\sigma \in M(H)$ is related to $F$ by

$$F(x) = \int_{H} \exp \{ i(h, x)^\sim \} d\sigma(h), \quad x \in B. \tag{1.4}$$

Let $\mathbb{C}$, $\mathbb{C}_+$ and $\mathbb{C}_+^*$ denote the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with nonnegative real part, respectively.

Let $F$ be a $\mathbb{C}$-valued scale-invariant measurable function on $B$ such that

$$J(\lambda) = \int_{B} F(\lambda^{-1/2} x) d\nu(x) \tag{1.5}$$

exists as a finite number for all real $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic Wiener integral of $F$ over $B$ with parameter $\lambda$, and for $\lambda \in \mathbb{C}_+$ we write

$$\int_{B}^{\text{anw}} F(x) d\nu(x) = J^*(\lambda). \tag{1.6}$$

Let $F$ be a functional on $B$ such that $\int_{B}^{\text{anw}} F(x) d\nu(x)$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists for nonzero real $q$, then we call it the analytic Feynman integral of $F$ over $B$ with parameter $q$ and we write

$$\int_{B}^{\text{anf}} F(x) d\nu(x) = \lim_{\lambda \to -iq} \int_{B}^{\text{anw}} F(x) d\nu(x) \tag{1.7}$$

where $\lambda \to -iq$ through $\mathbb{C}_+$.

**Notation.**

(i) For $\lambda \in \mathbb{C}_+$ and $y \in B$, let

$$\left( T_\lambda(F) \right)(y) = \int_{B}^{\text{anw}} F(x + y) d\nu(x). \tag{1.8}$$
(ii) Given a number $p$ with $1 \leq p < \infty$, $p$ and $p'$ will always be related by $\frac{1}{p} + \frac{1}{p'} = 1$.

(iii) Let $1 < p < \infty$ and let $G_n$ and $G$ be scale-invariant measurable functionals such that, for each $\alpha > 0$,

$$\lim_{n \to \infty} \int_B |G_n(\alpha x) - G(\alpha x)|^{p'} d\nu(x) = 0.$$  

Then we write

$$\lim_{n \to \infty} (w_s^{p'}) (G_n) \approx G$$

and call $G$ the scale-invariant limit in the mean of order $p'$. A similar definition is understood when $n$ is replaced by a continuously varying parameter.

**Definition 1.1.** Let $q \neq 0$ be a real number. For $1 < p < \infty$, we define the $L_p$ analytic Fourier-Feynman transform $T_q^{(p)}(F)$ of $F$ on $B$ by the formula ($\lambda \in \mathbb{C}_+$)

$$\lambda \to -iq$$

whenever this limit exists.

We define the $L_1$ analytic Fourier-Feynman transform $T_q^{(1)}(F)$ of $F$ by ($\lambda \in \mathbb{C}_+$)

$$\lambda \to -iq$$

for $s$-a.e. $y \in B$ whenever this limit exists.

In particular, we set

$$\lim_{\lambda \to -iq} (T_{\lambda}(F))(y) = \lim_{\lambda \to -iq} (T_{\lambda}(F))(y)$$

for $s$-a.e. $y \in B$ whenever this limit exists.

We note that, for $1 \leq p < \infty$, $T_q^{(p)}(F)$ is defined only $s$-a.e.. We also note that if $T_q^{(p)}(F_1)$ exists and if $F_1 \approx F_2$, then $T_q^{(p)}(F_2)$ exists and $T_q^{(p)}(F_1) \approx T_q^{(p)}(F_2)$.  

$$\int_B F(x) d\nu(x), \quad 1 \leq p < \infty.$$
2. The Wiener integral of variations of functionals

In this section, we obtain a basic theorem which expresses the analytic Feynman integral of the first variation of a functional $F$ in terms of the analytic Feynman integral of the product of $F$ with a linear factor.

**Definition 2.1.** Let $F$ be a Wiener measurable functional on $B$ and let $w \in B$. Then

$$\delta F(x|w) = \frac{\partial}{\partial t} F(x + tw)|_{t=0}$$

(if it exists) is called the first variation of $F(x)$ in the direction $w$.

The following theorem expresses the Wiener integral of the first variation of a functional $F$ in terms of the Wiener integral of the product of $F$ with a linear factor.

**Theorem 2.2.** Let $(H,B,\nu)$ be an abstract Wiener space and let $w \in H$. Let $F(x)$ be a Wiener integrable functional on $B$ and let $F(x)$ have the first variation $\delta F(x|w)$ for $x \in B$. Suppose that there exists a Wiener integrable functional $G(x)$ such that for some positive $\eta$,

$$\sup_{|t| \leq \eta} |\delta F(x + tw|w)| \leq G(x),$$

(2.15)

then both members of following equation exist and they are equal:

$$\int_B \delta F(x|w)d\nu(x) = \int_B F(x)((w,x)^\sim)d\nu(x).$$

(2.16)

**Proof.** We note that

$$\delta F(x + tw|w) = \frac{\partial}{\partial \lambda} F(x + tw + \lambda w)|_{\lambda=0}$$

(2.17)

$$= \frac{\partial}{\partial \mu} F(x + \mu w)|_{\mu=t}$$

$$= \frac{\partial}{\partial t} F(x + tw)$$

and since the first member of this equation exists, so does the last. By the mean value theorem, we obtain $F(x + tw) = F(x) + t\delta F(x + \theta tw|w)$ for some $\theta$ in $0 < \theta < 1$ depending on $t$. Hence it follows from the integrability of (2.15) and of $F(x)$ that

$$\sup_{|t| \leq \eta} |F(x + tw)|$$

(2.18)
is integrable on $B$. Now for $|t| \leq \eta$, we have the Cameron-Martin translation theorem in [16]

$$\int_B F(x) d\nu(x) = \exp\{-\frac{1}{2}t^2|w|^2\} \cdot \int_B F(x + tw) \exp\{-t(w, x)\} d\nu(x). \quad (2.19)$$

Differentiating formally with respect to $t$ and the setting $t = 0$, we obtain

$$\int_B \delta F(x|w) d\nu(x) = \int_B F(x)((w, x)^{\sim}) d\nu(x). \quad (2.20)$$

To justify this differentiation under the integral sign, we must show that

$$\sup_{|t| \leq m} |\delta F(x + tw|w) - F(x + tw) \exp\{-t(w, x)\} (w, x)^{\sim}|$$

is Wiener integrable on $B$ for some $\eta_1 > 0$. But it follows from the integrability of (2.18) that for some $\eta_2 > 0$

$$\sup_{|t| \leq \eta_2} |\delta F(x + tw|w)\exp\{\eta_1((w, x)^{\sim})\}|$$

is Wiener integrable on $B$. Similarly it follows from the integrability of (2.18) on $B$ that for some $\eta_3 > 0$

$$\sup_{|t| \leq \eta_3} |F(x + tw)\exp\{\eta_3((w, x)^{\sim})\}((w, x)^{\sim})|$$

is Wiener integrable on $B$. Taking $\eta_1 = \min\{\eta_2, \eta_3\}$, we obtain the Wiener integrability of (2.21) on $B$. Thus the theorem is established. \qed

**Corollary 2.3.** Let $(H, B, \nu)$ be an abstract Wiener space and let $w \in H$. For every $\rho > 0$ let $F(\rho x)$ be Wiener integrable on $B$. If $F(\rho x)$ have the first variation $\delta F(\rho x|\rho w)$ for all $x$ in $B$. Suppose that there exists a Wiener integrable functional $G(x)$ such that for some positive function $\eta(\rho)$

$$\sup_{|t| \leq \eta(\rho)} |\delta F(\rho x + \rho tw|\rho w)| \leq G(x), \quad (2.24)$$

then

$$\int_B \delta F(\rho x|\rho w) d\nu(x) = \int_B F(\rho x)((w, x)^{\sim}) d\nu(x). \quad (2.25)$$

**Proof.** We apply Theorem 2.2 to the functional after a change of scale. To do this we set

$$H(x) = F(\rho x)$$
and note that
\[ H(x + tw) = F(\rho x + t\rho w) \]
and
\[ \frac{\partial}{\partial t} H(x + tw)|_{t=0} = \frac{\partial}{\partial t} F(\rho x + t\rho w)|_{t=0} \]
or
\[ \delta H(x|w) = \delta F(\rho x|\rho w) \]
and the existence of either member implies that of the other.

Our next basic theorem expresses the analytic Feynman integral of the first variation of a functional \( F \) in terms of analytic Feynman integral of the product of \( F \) with a linear factor.

**Theorem 2.4.** Let \((H, B, \nu)\) be an abstract Wiener space and let \( w \in H \). For every \( \rho > 0 \) let \( F(\rho x) \) be Wiener integrable on \( B \). Let \( F(\rho x) \) have the first variation \( \delta F(\rho x|\rho w) \) for all \( x \) in \( B \). Suppose that there exists Wiener integrable \( G(x) \) such that for some positive function \( \eta(\rho) \),
\[
\sup_{|t| \leq \eta(\rho)} |\delta F(\rho x + \rho tw|\rho w)| \leq G(x),
\]
then if either member of the following equation exists, both analytic Feynman integrals below exist, and for each \( q(\neq 0) \in \mathbb{R} \)
\[
\int_B \alpha F(x|w) d\nu(x) = -iq \int_B \alpha F(x)[(w, x)^\gamma] d\nu(x).
\]

**Proof.** Let \( \rho \) be positive and set \( z = \frac{w}{\rho} \). Then using (2.25), we have
\[
\int_B \delta F(\rho x|w) d\nu(x) = \int_B \delta F(\rho x|\rho z) d\nu(x)
\]
\[
= \int_B F(\rho x)[(z, x)^\gamma] d\nu(x)
\]
\[
= \rho^{-2} \int_B F(\rho x)[(w, \rho x)^\gamma] d\nu(x).
\]
If we let \( \rho = \lambda^{-\frac{1}{2}} \), (2.28) becomes
\[
\int_B \delta F(\lambda^{-\frac{1}{2}} x|w) d\nu(x) = \lambda \int_B F(\lambda^{-\frac{1}{2}} x)[(w, \lambda^{-\frac{1}{2}} x)^\gamma] d\nu(x).
\]
Thus by the definition of the analytic Wiener integral, if either side of the following equation exists, then both exist and we have
\[
\int_B^{\text{anw}} \lambda B \delta F(x|w) d\nu(x) = \lambda \int_B^{\text{anw}} F(x)(w, x)^\sim d\nu(x).
\]
(2.30)
Letting \( \lambda \to -iq \) through \( \mathbb{C}_+ \), we have
\[
\int_B^{\text{anf}_q} \delta F(x|w) d\nu(x) = -iq \int_B^{\text{anf}_q} F(x)(w, x)^\sim d\nu(x).
\]
(2.31)

3. Integration by parts formula

In this section we obtain an integration by parts formula for analytic Feynman integrals and for Fourier-Feynman transform. We first state several facts.

(i) Let \( F \) and \( G \) be in \( \mathcal{F}(\mathbb{B}) \) with associated measures \( f \) and \( g \) respectively. Then, as was shown in [14, 15], their product \( K = FG \) is in \( \mathcal{F}(\mathbb{B}) \) with associated measure \( k \) satisfying
\[
\|k\| \leq \|f\| \|g\| \text{ where } \|\cdot\| \text{ is the total variation over } \mathbb{H}.
\]
In [8, 9], Chang, Kim and Yoo obtained following facts for the Fourier-Feynman transform and the first variation on \( \mathbb{B} \).

(ii) Let \( F \) be in \( \mathcal{F}(\mathbb{B}) \) with associated measure \( f \). Then, for all \( p \) with \( 1 \leq p < \infty \), the Fourier-Feynman transform \( T_q^{(p)}(F) \) exists for all \( q \in \mathbb{R} \setminus \{0\} \) and is given by the formula
\[
(T_q^{(p)}(F))(y) = \int_{\mathbb{H}} \exp\{i(h, y)^\sim - \frac{i}{2q} |h|^2\} df(h)
\]
\[
= \int_{\mathbb{H}} \exp\{i(h, y)^\sim\} d\mu(h)
\]
for s.a.e. \( y \) in \( \mathbb{B} \) where \( \mu \) is a complex Borel measure on \( \mathbb{H} \) defined by
\[
\mu(E) = \int_E \exp\{-\frac{i}{2q} |h|^2\} df(h)
\]
for every Borel set \( E \) in \( \mathbb{H} \), and so \( \|\mu\| \leq \|f\| \).

(iii) Let \( F \in \mathcal{F}(\mathbb{B}) \) so that
\[
F(x) = \int_{\mathbb{H}} \exp\{i(h, x)^\sim\} df(h)
\]
where \( f \) satisfies the condition \( \int_H |h| df(h) < \infty \). Then for each \( w \in \mathbb{H} \) and for s.a.e. \( y \in \mathbb{B} \), the first variation of \( F \), \( \delta F(y|w) \) is in \( \mathcal{F}(\mathbb{B}) \) and is
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given by the formula

\begin{equation}
\delta F(y|w) = \int_{H} i\langle h, w \rangle \exp \{i(h, y)\} df(h) \\
= \int_{H} \exp \{i(h, y)\} df_{w}(h)
\end{equation}

where \( f_{w}(E) \equiv \int_{E} i\langle h, w \rangle df(h) \), \( E \in \mathcal{B}(H) \), and so

\[ \|f_{w}\| \leq |w| \int_{H} |h||df(h)| < \infty. \]

(iv) Let \( F \) and \( G \) be elements of \( \mathcal{F}(\mathcal{B}) \) with associated measures \( f \) and \( g \) respectively, where \( f \) and \( g \) satisfy

\[ \int_{H} |h||df(h)| + |dg(h)| < \infty. \]

For each \( w \in H \),

\[ F(x)\delta G(x|w) + \delta F(x|w)G(x) \]

is an element of \( \mathcal{F}(\mathcal{B}) \).

(v) Let \( F \) be given as in (iv) and let \( 1 \leq p < \infty \) and \( q \in \mathbb{R} - \{0\} \). Then for each \( w \in H \) and for \( s \)-a.e. \( y \in \mathcal{B} \),

\begin{equation}
T_{q}^{(p)}(\delta F(\cdot|w))(y) = \delta T_{q}^{(p)}(F)(y|w) \\
= \int_{H} i\langle h, w \rangle \exp \left\{ i(h, y) - \frac{i}{2q} |h|^{2} \right\} df(h).
\end{equation}

In the following theorem, we obtain an integration by parts formula for analytic Feynman integral over \( \mathcal{B} \).

**Theorem 3.1.** Let \( F, G, f, g \) and \( w \) be given as (iv) above. Then for all \( q \in \mathbb{R} - \{0\} \),

\begin{equation}
\int_{B} \text{anf}_{q} [F(x)\delta G(x|w) + \delta F(x|w)G(x)]d\nu(x) \\
= -iq \int_{B} \text{anf}_{q} F(x)G(x)(w, x) d\nu(x).
\end{equation}
Proof. Let $K(x) = F(x)G(x)$. Then for all $\rho > 0$ and $t \in \mathbb{R}$,

$$
\begin{align*}
|\delta K(\rho x + \rho tw)| & = |F(\rho x + \rho tw)\delta G(\rho x + \rho tw)| \\
& = |F(\rho x + \rho tw)\delta G(\rho x + \rho tw)| \\
& \leq \rho |\|f||w| \int_{H} |h||dg(h)|| + \rho ||g||w| \int_{H} |h||df(h)|
\end{align*}
$$

and the last member of the above expression is Wiener integrable in $x$ for all $\rho > 0$. Also $K(x)$ is Wiener integrable and so by Theorem 2.4, stated in Section 2, equation (3.36) holds for all $q \in \mathbb{R} - \{0\}$. \hfill \square

The following integration by parts formula for Fourier-Feynman transform follows from (i)~(v) and Theorem 3.1.

**Theorem 3.2.** Let $F, G, f, g$ and $w$ be given as in Theorem 3.1. Then for $1 \leq p < \infty$ and $q \in \mathbb{R} - \{0\}$

$$
\begin{align*}
\int_{B}^{\text{anf}_{q}} [T_{q}^{(p)}(F)(x)\delta T_{q}^{(p)}(G)(x|w) + \delta T_{q}^{(p)}(F)(x|w)T_{q}^{(p)}(G)(x)]\nu(x)
\end{align*}
$$

$$
= -iq \int_{B}^{\text{anf}_{q}} T_{q}^{(p)}(F)(x)T_{q}^{(p)}(G)(x)[(w,x)^{\sim}]\nu(x).
$$

4. Transforms of functionals in $\mathcal{F}(B)$ multiplied with $n$ linear factors

In this section we establish the Fourier-Feynman transform of functionals of the form

$$
F_{n}(x) = F(x) \prod_{j=1}^{n}(w_{j}, x)^{\sim}
$$

with $F \in \mathcal{F}(B)$ and each $w_{j} \in H$.

We will show that the condition

$$
\int_{H} |h||n|df(h)| < \infty \tag{4.40}
$$

will ensure the existence of $T_{q}^{(p)}(F_{n})(y)$ for $s$-a.e. $y \in B$. In addition, since

(4.40) implies that

$$
\int_{H} |h||k|df(h)| < \infty \tag{4.41}
$$
for $k = 1, \cdots, n - 1$, condition (4.40) will also ensure the existence of $T^{(p)}_q(F_k)$ for $k = 1, \cdots, n - 1$.

The next theorem gives a recurrence relation in which we express the transform of $F_k$ in terms of the transforms and variation of $F_{k-1}$.

**Theorem 4.1.** Assume that $T^{(p)}_q(\delta F_{k-1}(\cdot|w_k))(y) = \delta T^{(p)}_q(F_{k-1})(y|w_k)$ exists for s.a.e. $y \in B$. Then $T^{(p)}_q(F_k)(y)$ exists for s.a.e. $y \in B$ and is given by the recurrence relation

\[
T^{(p)}_q(F_k)(y) = \left( \frac{i}{q} \right) T^{(p)}_q(\delta F_{k-1}(\cdot|w_k))(y) + (w_k, y)^\sim T^{(p)}_q(F_{k-1})(y).
\]

**Proof.** Since $T^{(p)}_q(\delta F_{k-1}(\cdot|w_k))(y)$ exists, we know that $\delta F_{k-1}(\rho x + y|w_k)$ is Wiener integrable for each $\rho > 0$ and hence by Theorem 2.4,

\[
\begin{align*}
T^{(p)}_q(F_k)(y) &= \int_B T^{(p)}_q(\delta F_{k-1}(\cdot|w_k))(x) d\nu(x) \\
&= \left( \frac{i}{q} \right) \int_B F_{k-1}(x + y)(w_k, x + y)^\sim d\nu(x) + i q \int_B F_{k-1}(x + y)(w_k, y)^\sim d\nu(x) \\
&= -i q \int_B F_k(x + y) d\nu(x) + i q(w_k, y)^\sim \int_B F_{k-1}(x + y) d\nu(x) \\
&= -i q T^{(p)}_q(F_k)(y) + i q(w_k, y)^\sim T^{(p)}_q(F_{k-1})(y).
\end{align*}
\]

Now solving (4.43) for $T^{(p)}_q(F_k)(y)$ yields (4.42) as desired. \qed

Our next result, which follows from Theorem 4.1 gives a recurrence relation for $T^{(p)}_q(F_k)(y) = \delta T^{(p)}_q(F_{k-1})(y|w_{k+1})$.

**Theorem 4.2.** Assume that

\[
\delta^2 T^{(p)}_q(F_{k-1})(\cdot|w_k)(y|w_{k+1}) = \delta T^{(p)}_q(\delta F_{k-1}(\cdot|w_k))(y|w_{k+1})
\]
exists for s.a.e. \( y \in B \). Then \( T_q^{(p)}(\delta F_k(\cdot|w_{k+1}))(y) \) exists for s.a.e. \( y \in B \) and is given by the recurrence relation

\[
(4.45) T_q^{(p)}(\delta F_k(\cdot|w_{k+1}))(y) = \left( \frac{i}{q} \right) \delta T_q^{(p)}(\delta F_{k-1}(\cdot|w_k))(y|w_{k+1}) \\
+ (w_k, w_{k+1}) T_q^{(p)}(F_{k-1})(y) \\
+ (w_k, y)^{-\delta T_q^{(p)}(\delta F_{k-1}(\cdot|w_{k+1}))(y)}.
\]

Next we will use Theorem 4.1 and Theorem 4.2 to establish that equation (4.42) is valid for \( k = 1, 2, \ldots, n \) where of course \( F_0 = F \).

First, for \( F \in \mathcal{F}(B) \) assume that its associated measure \( f \) satisfies

\[
\int_H |h|^2 df(h) < \infty.
\]

Then by (ii) and (iii) in Section 3 above, we see that \( \delta F(y|w_1) \) and \( T_q^{(p)}(\delta F(\cdot|w_1))(y) = \delta T_q^{(p)}(F)(y|w_1) \) are in \( \mathcal{F}(B) \). A direct calculation shows that

\[
(4.46) \delta T_q^{(p)}(F)(y|w_1) = \int_H i\langle h, w_1 \rangle \exp \left\{ i\langle h, y \rangle - \frac{i}{2q} |h|^2 \right\} df(h)
\]

holds for s.a.e. \( y \in B \). Hence using Theorem 4.1 with \( k = 1 \), we see that

\[
(4.47) T_q^{(p)}(F_1)(y) = \left( \frac{i}{q} \right) \delta T_q^{(p)}(F)(y|w_1) + (w_1, y)^{-\delta T_q^{(p)}(F)(y)}
\]

for s.a.e. \( y \in B \).

Next assume that \( f \), the associated measure \( F \in \mathcal{F}(B) \), satisfies

\[
\int_H |h|^2 df(h) < \infty.
\]

We see that

\[
(4.48) \delta^2 T_q^{(p)}(F)(\cdot|w_1)(y|w_2) = -\int_H \langle h, w_1 \rangle \langle h, w_2 \rangle \exp \left\{ i\langle h, y \rangle - \frac{i}{2q} |h|^2 \right\} df(h)
\]

for s.a.e. \( y \in B \). In addition \( \delta^2 T_q^{(p)}(F) \) is in \( \mathcal{F}(B) \) and so by equation (4.45),

\[
(4.49) \delta T_q^{(p)}(F_1)(y|w_2) = T_q^{(p)}(\delta F_1(\cdot|w_2))(y) \\
= \left( \frac{i}{q} \right) \delta^2 T_q^{(p)}(F)(\cdot|w_1)(y|w_2) + (w_1, w_2) T_q^{(p)}(F)(y) \\
+ (w_1, y)^{-\delta T_q^{(p)}(F)(y|w_2)}
\]
for s-a.e. \( y \in B \). Hence using Theorem 4.1 with \( k = 2 \), we see that

\[
(4.50) \quad T_q^{(p)}(F_2)(y) = \left(\frac{i}{q}\right)\delta T_q^{(p)}(F_1)(y|w_2) + (w_2, y)^\sim T_q^{(p)}(F_1)(y).
\]

for s-a.e. \( y \in B \).

Continuing in this manner, we see that if \( F \in \mathcal{F}(B) \), satisfies \( \int_H |h|^n|df(h)| < \infty \), then

\[
(4.51) \quad \delta^nT_q^{(p)}(F)(\cdot|w_1)\cdots(\cdot|w_{n-1})(y|w_n)
\]

\[
= \int_H \left(\prod_{j=1}^n i\langle h, w_j \rangle\right) \exp\left\{i(h, y)^\sim - \frac{i}{2q}|h|^2\right\} df(h)
\]

for s-a.e. \( y \in B \). In addition, \( \delta^nT_q^{(p)}(F) \) is in \( \mathcal{F}(B) \) with associated measure \( \mu \) satisfying

\[
||\mu|| \leq \left(\prod_{j=1}^n |w_j|\right) \int_H |h|^n|df(h)| < \infty.
\]

Hence \( \delta T_q^{(p)}(F_{n-1}(y|w_n)) \) exists for s-a.e. \( y \in B \) and is given by

\[
(4.52) \quad \delta T_q^{(p)}(F_{n-1})(y|w_n)
= \delta T_q^{(p)}(F_{n-1}(\cdot|w_n))(y)
= \left(\frac{i}{q}\right)^0 \left[\langle w_{n-1}, w_n \rangle T_q^{(p)}(F_{n-2})(y) + (w_{n-1}, y)^\sim \delta T_q^{(p)}(F_{n-2})(y|w_n)\right]
+ \left(\frac{i}{q}\right)^1 \left[\langle w_{n-2}, w_{n-1} \rangle \delta T_q^{(p)}(F_{n-3})(y|w_n) + \langle w_{n-2}, w_n \rangle \right.
\]

\[
\times \delta T_q^{(p)}(F_{n-3})(\cdot|w_{n-1}) + (w_{n-2}, y)^\sim \delta^2 T_q^{(p)}(F_{n-3})(\cdot|w_{n-1})(y|w_n)
\]

\[
+ \left(\frac{i}{q}\right)^2 \left[\langle w_{n-3}, w_{n-2} \rangle \delta^2 T_q^{(p)}(F_{n-4})(\cdot|w_{n-1}) + \langle w_{n-3}, w_{n-1} \rangle \delta^2 T_q^{(p)}(F_{n-4})(\cdot|w_{n-2}) + \langle w_{n-3}, y \rangle^\sim \delta^3 T_q^{(p)}(F_{n-4})(\cdot|w_{n-2})(\cdot|w_{n-1})(y|w_n)\right]
\]

\[
+ \cdots + \left(\frac{i}{q}\right)^{n-2} \left[\langle w_1, w_2 \rangle \delta^{n-2} T_q^{(p)}(F)(\cdot|w_3)\cdots(\cdot|w_{n-1})(y|w_n)\right.
\]

\[
+ \langle w_1, w_3 \rangle \delta^{n-2} T_q^{(p)}(F)(\cdot|w_2)\cdots(\cdot|w_{n-1})(y|w_n)
\]

\[
+ \cdots + \langle w_1, w_n \rangle \delta^{n-2} T_q^{(p)}(F)(\cdot|w_2)\cdots(\cdot|w_{n-1})(y|w_n)
\]

\[
+ (w_1, y)^\sim \delta^{n-1} T_q^{(p)}(\cdot|w_2)\cdots(\cdot|w_{n-1})(y|w_n)\right].
\]
Thus by Theorem 4.1 with \( k = n \), we obtain that
\[
T^q_p(F_n)(y) = \left( i \right)^n \delta^n T^q_p(F)(\cdot|w_1) \cdots (\cdot|w_n)(y|w_n).
\]

Therefore, by Theorem 4.3, we have
\[
T^q_p(F_n)(y) = \left( i \right)^n \delta^n T^q_p(F)(\cdot|w_1) \cdots (\cdot|w_n)(y|w_n) + (w_n, y) \sim T^q_p(F_n-1)(y)
\]
for s.a.e. \( y \in B \).

**Theorem 4.3.** Let \( F_n(x) = F(x) \prod_{j=1}^n (w_j, x) \sim \) with \( F \in \mathcal{F}(B) \) whose associated measure \( f \) satisfies \( \int_{\mathcal{H}} |h|^n |df(h)| < \infty \). Then for \( k = 1, 2, \ldots, n \),
\[
T^q_p(F_k)(y) = \left( i \right)^k \delta^k T^q_p(F)(\cdot|w_1) \cdots (\cdot|w_k)(y|w_k)
\]
for s.a.e. \( y \in B \).

Next, for special cases \( n = 1, 2 \) and \( 3 \), we express \( T^q_p(F_1), T^q_p(F_2) \)
and \( T^q_p(F_3) \) in terms of \( T^q_p(F), \delta T^q_p(F), \delta^2 T^q_p(F) \) and \( \delta^3 T^q_p(F) \).

\[
T^q_p(F_1)(y) = \left( i \right)^1 \delta T^q_p(F)(\cdot|w_1)(y|w_1) + (w_1, y) \sim T^q_p(F)(y).
\]

\[
T^q_p(F_2)(y) = \left( i \right)^2 \delta^2 T^q_p(F)(\cdot|w_1)(\cdot|w_2)(y|w_2) + \left( i \right)^2 \delta T^q_p(F)(\cdot|w_1)(y|w_2)
\]
\[
+ (w_2, y) \sim T^q_p(F)(\cdot|w_1) + (w_1, w_2) T^q_p(F)(y)
\]
\[
+ (w_1, y) \sim (w_2, y) \sim T^q_p(F)(y).
\]

\[
T^q_p(F_3)(y) = \left( i \right)^3 \delta^3 T^q_p(F)(\cdot|w_1)(\cdot|w_2)(\cdot|w_3)
\]
\[
+ \left( i \right)^2 \delta^2 T^q_p(F)(\cdot|w_2)(y|w_3)
\]
\[
+ (w_2, y) \sim \delta^2 T^q_p(F)(\cdot|w_1)(y|w_3)
\]
\[
+ (w_3, y) \sim \delta^2 T^q_p(F)(\cdot|w_1)(y|w_2)
\]
\[
+ (w_1, w_2) \delta T^q_p(F)(y|w_3) + (w_1, w_3) \delta T^q_p(F)(y|w_2)
\]
\[
+ \langle w_2, w_3 \rangle \delta T_q^{(p)}(F)(y|w_1) \\
+ \left( \frac{i}{q} \right) \left\{ T_q^{(p)}(F)(y) \right\} (w_1, y) \langle w_2, w_3 \rangle + (w_2, y) \langle w_1, w_3 \rangle \\
+ (w_3, y) \langle w_1, w_2 \rangle + (w_2, y) \langle w_3, y \rangle \delta T_q^{(p)}(F)(y|w_1) \\
+ (w_1, y) \langle w_3, y \rangle \delta T_q^{(p)}(F)(y|w_2) + (w_1, y) \langle w_2, y \rangle \\
\cdot \delta T_q^{(p)}(F)(y|w_3) \right\} + (w_1, y) \langle w_2, y \rangle \langle w_3, y \rangle \delta T_q^{(p)}(F)(y).
\]

Finally, setting \( y \equiv 0 \), we obtain the following Feynman integration formulas.

\[(4.58) \quad T_q^{(p)}(F_1)(0) = \int_B^{\text{ant}_q} F(x)(w_1, x) \sim (w_2, x) \sim (w_3, x) \sim d\nu(x) \]
\[
= \left( \frac{i}{q} \right) \int_H i \langle h, w_1 \rangle \exp \left\{ -\frac{i}{2q} |h|^2 \right\} df(h).
\]

\[(4.59) \quad T_q^{(p)}(F_2)(0) = \int_B^{\text{ant}_q} F(x)(w_1, x) \sim (w_2, x) \sim (w_3, x) \sim d\nu(x) \]
\[
= -\left( \frac{i}{q} \right)^2 \int_H \langle h, w_1 \rangle \langle h, w_2 \rangle \exp \left\{ -\frac{i}{2q} |h|^2 \right\} df(h) \\
+ \left( \frac{i}{q} \right) \langle w_1, w_2 \rangle \int_H \exp \left\{ -\frac{i}{2q} |h|^2 \right\} df(h).
\]

\[(4.60) \quad T_q^{(p)}(F_3)(0) = \int_B^{\text{ant}_q} F(x)(w_1, x) \sim (w_2, x) \sim (w_3, x) \sim d\nu(x) \]
\[
= -\left( \frac{i}{q} \right)^3 \int_H i \langle h, w_1 \rangle \langle h, w_2 \rangle \langle h, w_3 \rangle \exp \left\{ -\frac{i}{2q} |h|^2 \right\} df(h) \\
+ \left( \frac{i}{q} \right)^2 \int_H i \exp \left\{ -\frac{i}{2q} |h|^2 \right\} \left[ \langle w_2, w_3 \rangle \langle h, w_1 \rangle \\
+ \langle w_1, w_3 \rangle \langle h, w_2 \rangle + \langle w_1, w_2 \rangle \langle h, w_3 \rangle \right] df(h).
\]
By the way, if \( n = 4 \), we get the following analytic Feynman integration formula:

\[
(4.61) \quad T_q^{(p)}(F_4)(0) = \int_B \text{ant}_q \, F(x) \left( \prod_{j=1}^4 (w_j, x) \right) dv(x)
\]

\[
= \left( \frac{i}{q} \right)^4 \int_H \left( \prod_{j=1}^4 i(h, w_j) \right) \exp \left\{ - \frac{i}{2q} |h|^2 \right\} df(h)
\]

\[
+ \left( \frac{i}{q} \right)^3 \int_H \exp \left\{ - \frac{i}{2q} |h|^2 \right\} \left[ \langle w_1, w_2 \rangle \langle h, w_3 \rangle \langle h, w_4 \rangle - \langle w_1, w_3 \rangle \langle h, w_2 \rangle \langle h, w_4 \rangle - \langle w_1, w_4 \rangle \langle h, w_2 \rangle \langle h, w_3 \rangle
\]

\[
- \langle w_2, w_3 \rangle \langle h, w_1 \rangle \langle h, w_4 \rangle - \langle w_2, w_4 \rangle \langle h, w_1 \rangle \langle h, w_3 \rangle
\]

\[
- \langle w_3, w_4 \rangle \langle h, w_1 \rangle \langle h, w_2 \rangle \right] df(h) + \left( \frac{i}{q} \right)^2 \left[ \langle w_1, w_2 \rangle \langle w_3, w_4 \rangle
\]

\[
+ \langle w_1, w_3 \rangle \langle w_2, w_4 \rangle \langle w_1, w_4 \rangle \langle w_2, w_3 \rangle \right] \int_H \exp \left\{ - \frac{i}{2q} |h|^2 \right\} df(h).
\]

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