

# COMBINATORIAL PROOFS OF RAMANUJAN'S ${}_1\psi_1$ SUMMATION AND THE $q$ -GAUSS SUMMATION

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**Abstract.** Theorems in the theory of partitions are closely related to basic hypergeometric series. Some identities arising in basic hypergeometric series can be interpreted in the theory of partitions using F-partitions. In this paper, Ramanujan's  ${}_1\psi_1$  summation and the  $q$ -Gauss summation are established combinatorially.

## 1. INTRODUCTION

Georg Frobenius studied two-rowed arrays of distinct integers in decreasing order [12] in his work on group representation theory. His idea has been employed primarily in the representation theory of symmetric groups. In 1984, George E. Andrews [4] generalized the idea to the theory of partitions giving various restrictions on integers in two-rowed arrays. He called two-rowed arrays generalized Frobenius partitions or simply F-partitions. Some identities arising in  $q$ -series can be interpreted in the theory of partitions using F-partitions. In this paper, Ramanujan's  ${}_1\psi_1$  summation and the  $q$ -Gauss summation are combinatorially proved.

Recently, Sylvie Corteel and Jeremy Lovejoy have written a beautiful paper on overpartitions [9]. An overpartition of  $n$  is a partition of  $n$  where the last occurrence of a number may be overlined. Let  $\bar{p}(n)$  be the number of overpartitions of  $n$ . Then the generating function is

$$\sum_{n=0}^{\infty} \bar{p}(n)q^n = \prod_{i=1}^{\infty} \frac{1+q^i}{1-q^i} = 1 + \sum_{m=1}^{\infty} \frac{(1+1)(1+q)(1+q^{m-1})}{(1-q)(1-q^2)\cdots(1-q^m)} q^m \quad \text{for } |q| < 1, \quad (1.1)$$

where the second equality is obtained by the  $q$ -binomial theorem. There are combinatorial proofs of the second equality in (1.1) [9, 19, 21]. We describe another bijective proof of the second equality, which is a starting point of this work. We discuss this new bijection in Section 2.

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In the sequel, we assume that  $|q| < 1$  and use the customary notation for  $q$ -series

$$\begin{aligned} (a)_0 &:= (a; q)_0 := 1, \\ (a)_\infty &:= (a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k), \\ (a)_n &:= (a; q)_n := \frac{(a; q)_\infty}{(aq^n; q)_\infty} \quad \text{for any } n. \end{aligned}$$

The well known Ramanujan's  ${}_1\psi_1$  summation is given by

$$\sum_{n=-\infty}^{\infty} \frac{(a; q)_n}{(b; q)_n} z^n = \frac{(b/a; q)_\infty (q; q)_\infty (az; q)_\infty (q/az; q)_\infty}{(b; q)_\infty (q/a; q)_\infty (b/az; q)_\infty (z; q)_\infty} \quad \text{for } |b/a| < |z| < 1. \quad (1.2)$$

In the theory of  $q$ -series, the  ${}_1\psi_1$  summation is a central theorem. G. H. Hardy [15, pp. 222, 223] described it as a remarkable formula with many parameters. The first published proofs appear to be by W. Hahn [14] and M. Jackson [18] in 1949 and 1950, respectively. Jackson and her advisor W. M. Bailey [7] were the first to write it in the form (1.2). Other proofs have been given by C. Adiga, B. C. Berndt, S. Bhargava, and G. N. Watson [1], Andrews [2, 3], Andrews and R. Askey [5], Askey [6], M. Ismail [17], N. Fine [11], and K. Mimachi [20].

Recently, Corteel and Lovejoy [8] have given a combinatorial proof of the constant term, and Corteel [10] has completed their combinatorial proof using particle seas. However, her proof is not a bijection between two sets arising from both sides of the  ${}_1\psi_1$  summation. In Section 3, we establish a natural combinatorial proof. In fact, we give a second bijective proof, which is described in Section 5.

In the theory of basic hypergeometric series, the  $q$ -Gauss summation plays an important role. The  $q$ -Gauss summation [13] is given by

$$\sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(q; q)_n (c; q)_n} (c/ab)^n = \frac{(c/a; q)_\infty (c/b; q)_\infty}{(c; q)_\infty (c/ab; q)_\infty} \quad \text{for } |c/ab| < 1, \quad (1.3)$$

which was first proved by E. Heine [16]. The  $q$ -Gauss summation is normally proved by Heine's transformation formula. However, in this paper, we see that the  $q$ -Gauss summation is related to Ramanujan's  ${}_1\psi_1$  summation as mentioned and proved in [8, 10]. In Section 4, we explain  $q$ -Gauss summation in a combinatorial way using the bijection for the  ${}_1\psi_1$  summation.

## 2. F-PARTITIONS

For a nonnegative integer  $n$ , a Frobenius partition of  $n$  is a two-rowed array of row-wise distinct nonnegative integers

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

where each row is of the same length, each is arranged in decreasing order, and  $n = r + \sum_{i=1}^r (a_i + b_i)$ . A Frobenius partition is another representation of an ordinary partition. Putting  $r$  dots in the main diagonal direction and attaching  $a_i$  dots to the right of the

diagonal in rows and  $b_i$  dots below the diagonal in columns, we can obtain a Ferrers graph of an ordinary partition. For example, the Frobenius partition

$$\begin{pmatrix} 6 & 5 & 2 & 0 \\ 5 & 4 & 1 & 0 \end{pmatrix}$$

corresponds to the partition  $7 + 7 + 5 + 4 + 2 + 2$ , as seen easily from the Ferrers graph in Figure 1. In particular, Frobenius partitions can be used to prove Jacobi's triple product

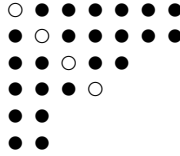


FIGURE 1. Ferrers graph of  $7 + 7 + 5 + 4 + 2 + 2$

identity [4, 22, 23]. Andrews has generalized Frobenius partitions in [4].

For sets  $A$  and  $B$  of certain kinds of partitions, we define

$$\mathcal{F}(A, B) = \left\{ \left( \begin{array}{cccc} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_s \end{array} \right) \mid (a_1 \ a_2 \ \cdots \ a_r) \in A, \ (b_1 \ b_2 \ \cdots \ b_s) \in B \right\}.$$

We call two-rowed arrays in  $\mathcal{F}(A, B)$  F-partitions. Let  $\phi_{A,B}(n)$  be the number of F-partitions in  $\mathcal{F}(A, B)$ , where the sum of parts in the top and bottom rows plus the number of parts in the top row equals  $n$ , and let  $\bar{\phi}_{A,B}(n)$  be the number of F-partitions counted by  $\phi_{A,B}(n)$  whose top and bottom rows have the same number of parts. Let  $\mathcal{D}$  be the set of all partitions into distinct nonnegative parts. Then, obviously  $\bar{\phi}_{\mathcal{D},\mathcal{D}}(n)$  is equal to  $p(n)$ , the number of partitions of  $n$ .

Let  $\mathcal{O}$  be the set of all overpartitions where some parts may be 0. Then the generating function for F-partitions counted by  $\phi_{\mathcal{O},\mathcal{O}}(n)$  is

$$\sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \phi_{\mathcal{O},\mathcal{O}}(n; k) z^k q^n = \frac{(-zq; q)_{\infty} (-z^{-1}; q)_{\infty}}{(zq; q)_{\infty} (z^{-1}; q)_{\infty}}, \quad (2.1)$$

where  $\phi_{\mathcal{O},\mathcal{O}}(n; k)$  is the number of partitions counted by  $\phi_{\mathcal{O},\mathcal{O}}(n)$  such that  $k$  is equal to the number of parts in the top row minus the number of parts in the bottom row. The right side of (2.1) is related to Ramanujan's  ${}_1\psi_1$  summation formula, which is studied in Section 3.

On the other hand, the constant term of (2.1), which is the generating function for F-partitions counted by  $\bar{\phi}_{\mathcal{O},\mathcal{O}}(n)$ , can be written as

$$\sum_{n=0}^{\infty} \bar{\phi}_{\mathcal{O},\mathcal{O}}(n) q^n = 1 + \sum_{n=1}^{\infty} \frac{(1+1)^2 (1+q)^2 \cdots (1+q^{n-1})^2}{(q; q)_n^2} q^n = \frac{(-q; q)_{\infty}^2}{(q; q)_{\infty}^2}, \quad (2.2)$$

where the first equality is obtained by applying Theorem 2.1, which is proved in this section and the last equality is obtained by applying the  $q$ -Gauss summation. We show the last equality of (2.2) by proving the  $q$ -Gauss summation in a combinatorial way in Section 4.

Recall that  $\bar{p}(n)$  is the number of overpartitions of  $n$  for which the generating function is given by (1.1). There is another representation of the generating function for overpartitions, which is stated in the following theorem.

**Theorem 2.1.** *Let  $\bar{p}(n; k)$  be the number of overpartitions of  $n$  into positive parts with  $k$  ordinary parts. Then*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \bar{p}(n; k) a^k q^n = 1 + \sum_{m=1}^{\infty} \frac{(a+1)(a+q)\cdots(a+q^{m-1})}{(q; q)_m} q^m. \quad (2.3)$$

*Proof.* We prove the theorem by showing that each term of the right side of (2.3) generates overpartitions into exactly  $m$  positive parts. First we see that  $q^m/(q; q)_m$  generates ordinary partitions  $\lambda = (\lambda_1 \cdots \lambda_m)$  into exactly  $m$  positive parts. On the other hand, let  $b = (b_1, \dots, b_m)$  be a bit string of length  $m$ . We define  $\mu = (\mu_1 \cdots \mu_m)$  by

$$\mu_i = \begin{cases} \lambda_i, & \text{if } b_i = 0, \\ \lambda_i + m - i, & \text{if } b_i = 1. \end{cases}$$

We obtain an overpartition  $\tilde{\mu}$  into exactly  $m$  parts rearranging  $\mu_i$  in weakly decreasing order.

This algorithm is invertible. For an overpartition  $\nu$ , we subtract  $m - j$  from  $\nu_j$  if  $\nu_j$  is overlined. Let the resulting sequence be  $c = (c_1, \dots, c_m)$ . If  $c_j < c_{j+1}$ , we switch  $\nu_j$  and  $\nu_{j+1}$ . Successively, we apply the process above to the  $\nu$  obtained in the previous step. Let  $\sigma$  be the resulting sequence. We produce a pair of a partition  $\tilde{\sigma}$  and a bit string  $d$ , where

$$\tilde{\sigma}_i = \begin{cases} \sigma_i, & \text{if } \sigma_i \text{ is unrestricted,} \\ \sigma_i - m + i, & \text{if } \sigma_i \text{ is overlined,} \end{cases}$$

$$d_i = \begin{cases} 0, & \text{if } \sigma_i \text{ is unrestricted,} \\ 1, & \text{if } \sigma_i \text{ is overlined.} \end{cases}$$

The process described above is the reverse of rearranging ordinary parts and overlined parts in weakly decreasing order.  $\square$

Similarly, we see that the generating function for overpartitions in  $\mathcal{O}$  can be written as

$$1 + \sum_{m=1}^{\infty} \frac{(a+1)(a+q)\cdots(a+q^{m-1})}{(q; q)_m}. \quad (2.4)$$

Throughout this paper, for an overpartition  $\alpha$  in  $\mathcal{O}$ , we use notation  $\{\alpha\}$  and  $[\alpha]$  for the array generated by  $(a+1)\cdots(a+q^{m-1})$  and the partition generated by  $1/(q; q)_m$ , respectively, in (2.4), and we say the parts  $\alpha_i$  of the overpartition  $\alpha$  are in overlined order when  $\alpha_i$  is equal to  $\{\alpha\}_i + [\alpha]_i$ , where  $\{\alpha\}_i$  and  $[\alpha]_i$  are the  $i$ th entries of  $\{\alpha\}$  and  $[\alpha]_i$ , respectively.

### 3. RAMANUJAN'S ${}_1\psi_1$ SUMMATION

To interpret Ramanujan's  ${}_1\psi_1$  summation in a combinatorial way, we need to make some changes of variables in (1.2). Substituting  $azq$ ,  $-a^{-1}$ , and  $-bq$  for  $z$ ,  $a$ , and  $b$ ,

respectively, we obtain

$$\begin{aligned} \frac{(-zq; q)_\infty (-z^{-1}; q)_\infty}{(azq; q)_\infty (bz^{-1}; q)_\infty} &= \frac{(-aq; q)_\infty}{(q; q)_\infty (abq; q)_\infty} \sum_{k=0}^{\infty} (-1/a; q)_k (-bq^{k+1}; q)_\infty a^k z^k q^k \\ &+ \frac{(-bq; q)_\infty}{(q; q)_\infty (abq; q)_\infty} \sum_{k=-1}^{-\infty} (-1/b; q)_{-k} (-aq^{-k+1}; q)_\infty b^{-k} z^k. \end{aligned} \quad (3.1)$$

When  $a = b = 1$ , the left side of (3.1) is the generating function for F-partitions counted by  $\phi_{\mathcal{O}, \mathcal{O}}(n)$  as seen in (2.1). On the other hand, for any  $k$ , each term on the right side of (3.1) can be interpreted as the generating function for quintuples: when  $k \geq 0$ ,

- two ordinary partitions,
- one partition into distinct parts,
- one partition into distinct parts greater than  $k$ ,
- one overpartition into  $k$  positive parts, where unrestricted parts are all 1's, and overlined parts are positive and less than or equal to  $k$ ,

and when  $k < 0$ ,

- two ordinary partitions,
- one partition into distinct parts,
- one partition into distinct parts greater than  $k$ ,
- one overpartition into  $k$  nonnegative parts, where unrestricted parts are all 0's, and overlined parts are nonnegative and less than  $k$ .

Let  $S(k)$  be the set of such quintuples for any integer  $k$ . Recall that  $\mathcal{F}(\mathcal{O}, \mathcal{O})$  is the set of F-partitions counted by  $\phi_{\mathcal{O}, \mathcal{O}}$ . We combinatorially interpret (3.1) in the next theorem.

**Theorem 3.1.** *There is a one to one correspondence between  $\mathcal{F}(\mathcal{O}, \mathcal{O})$  and  $\cup_{k=-\infty}^{\infty} S(k)$  such that a F-partition with the difference of the numbers of the top row and bottom row equal to  $k$  corresponds to a quintuple in  $S(k)$ .*

*Proof.* We rearrange parts of a given F-partition and then split the F-partition into five arrays. Note that the exponents of  $a$  and  $b$  in (3.1) indicate the numbers of unrestricted parts in the top row and the bottom row, respectively. Let an F-partition in  $\mathcal{F}(\mathcal{O}, \mathcal{O})$  be given by

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{r+k} \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}.$$

We consider two cases: when  $k \geq 0$  and  $k < 0$ .

Case I:  $k \geq 0$ .

1. Rearrange the parts in the top row in overlined order and denote the resulting array by

$$\alpha = (\alpha_1 \ \alpha_2 \ \cdots \ \alpha_{r+k}).$$

Rearrange the parts in the bottom row such that overlined parts follow unrestricted parts, overlined parts are in decreasing order, and unrestricted parts are in weakly increasing order. We denote the bottom row so obtained by

$$\beta = (\beta_1 \ \beta_2 \ \cdots \ \beta_r).$$

2. Let  $\mu$  be the array obtained by increasing each part of  $\{\alpha\}$  by 1, and let  $\lambda$  be  $[\alpha]$ . Produce two partitions into distinct parts  $\sigma, \nu$ , and two ordinary partitions  $\gamma, \delta$  as follows: for  $1 \leq i \leq r$ ,
  - put  $\beta_i + \mu_{r-i+1}$  as a part of  $\sigma$  if  $\beta_i$  is overlined and  $\mu_{r-i+1}$  is unrestricted,
  - put  $\beta_i + \mu_{r-i+1}$  as a part of  $\nu$  if  $\beta_i$  is unrestricted and  $\mu_{r-i+1}$  is overlined,
  - put  $\beta_i + \mu_{r-i+1}$  as a part of  $\gamma$  if  $\beta_i$  and  $\mu_{r-i+1}$  are overlined,
  - put  $\beta_i + \mu_{r-i+1}$  as a part of  $\delta$  if  $\beta_i$  and  $\mu_{r-i+1}$  are unrestricted.
 Note that the overlined parts of  $\beta$  and  $\mu$  are distinct. Thus,  $\sigma$  is a partition into distinct parts the number of which is counted by  $a$ ;  $\nu$  is the other partition into distinct parts the number of which is counted by  $b$ . Moreover, the parts of  $\nu$  are greater than  $k$  since  $\mu_{r-i+1}$  equals  $(k+i)$  if it is overlined. Note that the parts of  $\gamma$  are greater than or equal to  $(r+k)$  since  $\beta_i$  and  $\mu_{r-i+1}$  are greater than or equal to  $(r-i)$  and  $(k+i)$ , respectively, if they are overlined. Rearrange the parts of each partition in weakly decreasing order.
3. The remaining parts  $\mu_{r+1}, \dots, \mu_{r+k}$  of  $\mu$  form an overpartition  $\rho$  whose unrestricted parts are 1's, and overlined parts are positive and less than or equal to  $k$ .
4. Add the parts of the conjugate of  $\lambda$  to  $\gamma$  as parts. Note that conjugate of  $\lambda$  has parts less than or equal to  $(r+k)$ , since  $\lambda$  has at most  $(r+k)$  positive parts.

We see that the quintuple  $(\gamma, \delta, \sigma, \nu, \rho)$  is in  $S(k)$ .

The process above is reversible. Let a quintuple  $(\gamma, \delta, \sigma, \nu, \rho)$  in  $S(k)$  be given. Let  $l(c)$  be the number of entries in an array  $c$ . Note that the parts of a partition are in weakly decreasing order.

1. Make the parts of  $\nu$  overlined and decrease each part of  $\nu$  by  $(k+1)$ . Build an array  $\beta$  consisting of the parts of  $\delta$  and  $\nu$  such that parts in  $\beta$  are in overlined order. We use the inverse algorithm in the proof of Theorem 2.1. Increase each overlined part of  $\{\beta\}$  by  $(k+1)$ .
2. Make the parts of  $\gamma$  overlined. Build an array  $\alpha$  consisting of the parts of  $\sigma$  and  $\gamma$  as follows. Let  $\gamma$  be  $\alpha^{(0)}$ . We successively build arrays  $\alpha^{(i)}$  for  $1 \leq i \leq l(\sigma)$  in the following way. For brevity, we delete the superscripts  $(i)$ . For  $1 \leq j \leq l(\alpha)$ , do the following:
  - if  $\alpha_j$  is overlined and  $(\sigma_i + k + l(\beta) + j) \geq \alpha_j$ , then insert  $\sigma_i$  before  $\gamma_j$ ; stop;
  - otherwise, increase  $j$  by 1.

We denote the very last resulting array by  $\alpha$ .

3. Make two arrays  $\hat{\alpha}$  and  $\hat{\mu}$  as follows. For  $1 \leq i \leq l(\alpha)$ ,

$$\hat{\alpha}_i = \begin{cases} \overline{\alpha_i - 1}, & \text{if } \alpha_i \text{ is unrestricted,} \\ \overline{\alpha_i - (i + k + l(\beta))}, & \text{if } \alpha_i \text{ is overlined and } \bar{\alpha}_i - (i + k + l(\beta)) \geq 0, \end{cases}$$

$$\hat{\mu}_i = \begin{cases} 1, & \text{if } \alpha_i \text{ is unrestricted,} \\ \overline{i + k + l(\beta)}, & \text{if } \alpha_i \text{ is overlined and } \bar{\alpha}_i - (i + k + l(\beta)) \geq 0. \end{cases}$$

4. Write the parts of  $\hat{\mu}$  in reverse order, and attach the parts before the first part of  $\{\beta\}$  and the parts of  $\rho$  after the first part of  $\{\beta\}$  keeping the order of the parts of  $\hat{\mu}$  and  $\rho$ . We use  $\mu$  for the resulting array.

5. Take the conjugate of the partition consisting of the remaining parts of  $\alpha$  after deleting the bars. We denote the resulting partition by  $\lambda$ .
6. Decrease each part of  $\mu$  by 1. Add each part of  $\mu$  and  $\lambda$  to obtain an overpartition  $\tilde{\alpha}$ .
7. Add the parts of  $\hat{\alpha}$  as parts to  $[\beta]$  to obtain an overpartition  $\tilde{\beta}$ .
8. Combine  $\tilde{\alpha}$  and  $\tilde{\beta}$  to form a F-partition and rearrange the parts in nonincreasing order.

Case II:  $k < 0$ . By symmetry we can prove this case as we did with Case I. The only difference is that we should increase only the first  $(r + k)$  parts of  $\{\beta\}$  by 1. We omit the proof.  $\square$

**Example 1.** In this example, we show how to obtain two ordinary partitions  $\gamma, \delta$ , two partitions into distinct parts  $\sigma, \nu$ , and an array  $\rho$  using the bijection, and then show the reverse algorithm. Let

$$\begin{pmatrix} \bar{6} & \bar{5} & \bar{4} & 4 & 3 & 2 & \bar{0} \\ 6 & \bar{7} & 3 & \bar{3} & \bar{1} & & \end{pmatrix}$$

be a F-partition. We rearrange the top row and bottom row as in step 1 as follows.

$$\begin{pmatrix} \bar{6} & \bar{5} & \bar{4} & 4 & 3 & 2 & \bar{0} \\ 6 & \bar{7} & 3 & \bar{3} & \bar{1} & & \end{pmatrix} \longrightarrow \begin{pmatrix} 4 & 3 & \bar{6} & \bar{5} & \bar{4} & 2 & \bar{0} \\ 3 & 6 & \bar{7} & \bar{3} & \bar{1} & & \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Then,

$$\alpha = (4 \ 3 \ \bar{6} \ \bar{5} \ \bar{4} \ 2 \ \bar{0}) \longrightarrow [\alpha] = (4 \ 3 \ 2 \ 2 \ 2 \ 2 \ 0), \quad \{\alpha\} = (0 \ 0 \ \bar{4} \ \bar{3} \ \bar{2} \ 0 \ \bar{0}).$$

Increase each part of  $\{\alpha\}$  by 1 and denote the resulting array by  $\mu$ . The last two parts of  $\mu$  form an array  $\rho$ . Rearrange the remaining parts of  $\mu$  in reverse order, and add each part to the corresponding part of  $\beta$ .

$$\begin{pmatrix} \bar{3} & \bar{4} & \bar{5} & 1 & 1 \\ 3 & 6 & \bar{7} & \bar{3} & \bar{1} \\ \bar{6} & \bar{10} & \bar{12} & \bar{4} & \bar{2} \end{pmatrix}$$

Adding 12 to the conjugate of  $[\alpha]$  as a part, we obtain  $\gamma$ . Observe that  $\bar{6}$  and  $\bar{10}$  form a partition  $\nu$ , and  $\bar{4}, \bar{2}$  form a partition  $\sigma$ . Let  $\epsilon$  denote the partition with no parts.

$$\gamma = (12 \ 6 \ 6 \ 2 \ 1), \quad \delta = \epsilon, \quad \sigma = (\bar{4} \ \bar{2}), \quad \nu = (\bar{10} \ \bar{6}), \quad \rho = (1 \ \bar{1}),$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ ab & a & b \end{array}$$

In Figure 2, the dots to the right of the rightmost vertical line form  $[\alpha]$ , the dots above the diagonal form  $\{\alpha\}$ , and the dots below the diagonal form  $\beta$  in the bottom row of the given F-partition. The overlined parts of  $\{\alpha\}$  are marked by  $*$  on the dots on the diagonal.

We apply the reverse algorithm to  $(\gamma, \delta, \sigma, \nu, \rho)$ , where  $\gamma = (12 \ 6 \ 6 \ 2 \ 1)$ ,  $\delta = \epsilon$ ,  $\sigma = (4, 2)$ ,  $\nu = (10, 6)$  and  $\rho = (1, \bar{1})$ . Note that  $k = 2$  since  $\rho$  has two parts. To make  $\beta$ , we follow Step 1 with  $\nu$  and  $\delta$ . Since  $\delta = \epsilon$ ,  $\beta$  equals  $\nu$ , where  $\{\beta\} = (\bar{4}, \bar{3})$  and  $[\beta] = (6, 3)$ .

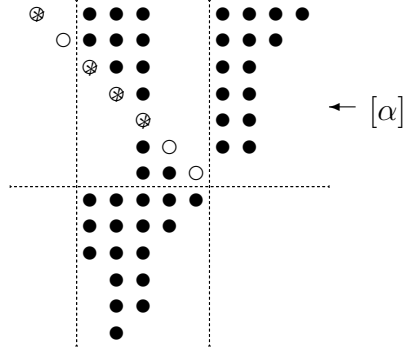


FIGURE 2

We take  $\sigma$  and  $\gamma$ , and apply Step 2 to make  $\alpha$ . Let  $\alpha^{(0)}$  be  $\gamma$ . We insert the parts of  $\sigma$  as follows. Since

$$\sigma_1 + k + l(\beta) + 1 = 4 + 2 + 2 + 1 < \alpha_1^{(0)} = \bar{1}2,$$

$$\sigma_1 + k + l(\beta) + 2 = 4 + 2 + 2 + 2 > \alpha_2^{(0)} = \bar{6},$$

insert  $\sigma_1$  between  $\alpha_1^{(0)}$  and  $\alpha_2^{(0)}$ . Let the resulting partition  $(\bar{1}2\ 4\ \bar{6}\ \bar{6}\ \bar{2}\ \bar{1})$  be  $\alpha^{(1)}$ . Since

$$\sigma_2 + k + l(\beta) + 1 = 2 + 2 + 2 + 1 < \alpha_1^{(1)} = \bar{1}2,$$

$$\sigma_2 + k + l(\beta) + 3 = 2 + 2 + 2 + 3 > \alpha_3^{(1)} = \bar{6},$$

insert  $\sigma_2$  between  $\alpha_2^{(1)}$  and  $\alpha_3^{(1)}$ . Since no parts of  $\sigma$  remain, we stop. Let  $\alpha$  be  $(\bar{1}2\ 4\ 2\ \bar{6}\ \bar{6}\ \bar{2}\ \bar{1})$ . By Step 3, we obtain  $\hat{\alpha} = (\bar{7}\ \bar{3}\ \bar{1})$  and  $\hat{\mu} = (\bar{5}\ 1\ 1)$ . By Step 4, we obtain  $\mu = (1\ 1\ \bar{5}\ \bar{4}\ \bar{3}\ 1\ \bar{1})$ . By Step 5, we obtain  $\lambda$ , where  $\{\lambda\} = \mu$  and  $[\lambda] = (4\ 3\ 2\ 2\ 2)$ . By Step 6, we obtain  $\tilde{\alpha} = (4\ 3\ \bar{6}\ \bar{5}\ \bar{4}\ 2\ \bar{0})$ . By Step 7, we obtain  $\tilde{\beta} = (\bar{6}\ \bar{7}\ \bar{3}\ \bar{3}\ \bar{1})$ . By combining  $\tilde{\alpha}$  and  $\tilde{\beta}$ , we obtain

$$\begin{pmatrix} \bar{6} & \bar{5} & \bar{4} & 4 & 3 & 2 & \bar{0} \\ 6 & \bar{7} & 3 & \bar{3} & \bar{1} & & \end{pmatrix}$$

#### 4. $q$ -GAUSS SUMMATION

Replacing  $a$ ,  $b$ , and  $c$  by  $a^{-1}$ ,  $b^{-1}$ , and  $cq$ , respectively, in the left side of (1.3), we obtain

$$\begin{aligned} & 1 + \sum_{n=1}^{\infty} \frac{(a+1)(a+q)\cdots(a+q^{n-1})(b+1)(b+q)\cdots(b+q^{n-1})}{(q;q)_n(cq;q)_n} (cq)^n \\ &= \frac{(-acq;q)_{\infty}(-bcq;q)_{\infty}}{(cq;q)_{\infty}(abcq;q)_{\infty}}, \end{aligned} \tag{4.1}$$

which coincides with (2.2) when  $a = b = c = 1$ . Thus the left side of (4.1) is the generating function for the F-partitions counted by  $\bar{\phi}_{\mathcal{O},\mathcal{O}}(n)$ , where the exponents of  $a$  and  $b$  are the number of unrestricted parts in the top row and the number of unrestricted parts in the bottom row, respectively. Therefore, (4.1) is a refinement of the constant term of Ramanujan's  ${}_1\psi_1$  summation (3.1). In this section, we explain how the bijection established in Section 3 can be applied to  $q$ -Gauss summation.



Let an F-partition counted by  $\bar{\phi}_{\mathcal{O},\mathcal{O}}(n)$  be given by

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}.$$

Rearranging the parts in the top row and dividing the resulting partition into  $\lambda$  and  $\mu$ , we see that the exponent of  $c$  in (4.1) equals  $(r + \lambda_1)$ . Since we add  $\beta_i$  and  $\mu_{r-i+1}$ , and take the conjugate of  $\lambda$ , we obtain exactly  $(r + \lambda_1)$  parts in total, which shows that the statistic tracked by  $c$  has been preserved under the bijection.

**Example 2.** Let

$$\begin{pmatrix} 4 & \bar{3} & 3 & 2 & \bar{0} \\ \bar{7} & 6 & \bar{3} & 3 & \bar{1} \end{pmatrix}$$

be an F-partition. As we did in Example 1, we rearrange the top row and bottom row, and then divide the top row into  $[\alpha]$  and  $\{\alpha\}$  as follows:

$$\begin{pmatrix} 4 & \bar{3} & 3 & 2 & \bar{0} \\ \bar{7} & 6 & \bar{3} & 3 & \bar{1} \end{pmatrix} \longrightarrow \begin{pmatrix} 4 & 3 & 2 & \bar{3} & \bar{0} \\ 3 & 6 & \bar{7} & \bar{3} & \bar{1} \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

$$(4 \ 3 \ 2 \ \bar{3} \ \bar{0}) \longrightarrow [\alpha] = (4 \ 3 \ 2 \ 2 \ 0), \quad \{\alpha\} = (0 \ 0 \ 0 \ \bar{1} \ \bar{0}).$$

After increasing each part of  $\{\alpha\}$  by 1 and rearranging them in reverse order, we add each part to the corresponding part of  $\beta$ .

$$\begin{pmatrix} \bar{1} & \bar{2} & 1 & 1 & 1 \\ 3 & 6 & \bar{7} & \bar{3} & \bar{1} \\ \bar{4} & \bar{8} & \bar{8} & \bar{4} & \bar{2} \end{pmatrix}$$

We obtain the following four partitions.

$$\begin{array}{cccc} (4 \ 4 \ 2 \ 1) & \epsilon & (\bar{8} \ \bar{4} \ \bar{2}) & (\bar{8} \ \bar{4}) \\ \uparrow & \uparrow & \uparrow & \uparrow \\ c & abc & ac & bc \end{array}$$

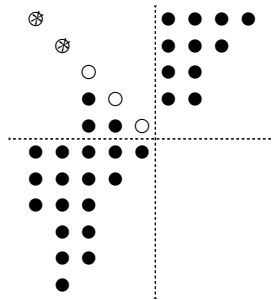


FIGURE 3

5. ANOTHER PROOF OF RAMANUJAN'S  ${}_1\psi_1$  SUMMATION

Cortee and Lovejoy [8] have established a combinatorial proof of the special case of Ramanujan's  ${}_1\psi_1$  summation when  $a = 0$ . In this section, we generalize their proof to establish Ramanujan's  ${}_1\psi_1$  summation for any  $a$ .

Let a F-partition counted by  $\phi_{\mathcal{O},\mathcal{O}}(n)$  be given by

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{r+k} \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}.$$

We consider two cases:  $k \geq 0$  and  $k < 0$ .

Case I:  $k \geq 0$ .

1. Rearrange the bottom row in the order that overlined parts follow unrestricted parts.
2. Shift the overlined parts in the bottom row to the right so that the last part is under  $a_{r+k}$ .
3. Divide the F-partition into two arrays;
  - F-partition with all the overlined parts in the bottom row and corresponding parts in the top row; call it  $A = \begin{pmatrix} c_1 & \cdots & c_t \\ d_1 & \cdots & d_t \end{pmatrix}$ .
  - F-partition with the remainders; call it  $B = \begin{pmatrix} \gamma_1 & \cdots & \gamma_u \\ \delta_1 & \cdots & \delta_v \end{pmatrix}$ .
4. Apply the following rule to  $A$ . We denote the pair of empty partitions  $(\epsilon, \epsilon)$  by  $(\alpha^{(0)}, \beta^{(0)})$  and successively build pairs of partitions  $(\alpha^{(i)}, \beta^{(i)})$  for  $1 \leq i \leq t$  in the following way. For brevity, we delete the superscripts  $(i)$ .
  - if  $c_{t-i+1}$  is overlined, then add  $d_{t-i+1}$  as a part to the conjugate of  $\alpha$ , take the conjugate of the resulting partition, and add  $(c_{t-i+1} + 1)$  as a part to the partition.
  - if  $c_{t-i+1}$  is unrestricted, then add  $c_{t-i+1}$  as a part to  $\alpha$  and add  $(d_{t-i+1} + 1)$  as a part to  $\beta$ .

Let the very last pair of partitions  $(\alpha^{(t)}, \beta^{(t)})$  be  $(\alpha, \beta)$ . Note that  $\beta$  is a partition into distinct parts the number of which is counted by  $a$ , since the  $c_i$ 's are unrestricted and the  $d_i$ 's are overlined, and  $\alpha$  is a partition into parts less than or equal to the smallest part  $\gamma_u$  in the top row of  $B$ , since the  $c_i$ 's are less than  $\gamma_u$  if overlined.

5. Rearrange the parts in the top row of  $B$  in overlined order. We denote the resulting overpartition by  $\sigma$ . Let  $\mu$  be the array obtained by increasing each part of  $\{\sigma\}$  by 1.
6. Produce a partition  $\nu$  into distinct parts and an ordinary partition  $\pi$  as follows: for  $1 \leq i \leq v$ ,
  - put  $\delta_i + \mu_i$  as a part of  $\nu$  if  $\mu_i$  is overlined,
  - put  $\delta_i + \mu_i$  as a part of  $\pi$  if  $\mu_i$  is unrestricted.

Note that the overlined parts of  $\mu$  are distinct and in decreasing order. Thus,  $\nu$  is a partition into distinct parts the number of which is counted by  $b$ . The parts of  $\nu$  are greater than  $(u - v)$ , which equals  $k$  since  $\mu_v$  is  $(u - v + 1)$  if it is overlined. We denote the partition consisting of the remaining parts of  $\nu$  by  $\rho$ . Note that  $\rho$  is a  $k$ -tuple of 1's and overlined integers less than or equal to  $k$ .

7. Add the parts of  $[\sigma]$  to the partition  $\alpha$  obtained in step 4. Note that the parts of  $[\sigma]$  are greater than or equal to the parts of  $\alpha$ .
8. We obtain a quintuple  $(\alpha, \beta, \nu, \pi, \rho)$ , where  $\alpha$  and  $\pi$  are ordinary partitions,  $\beta, \nu$  are two partitions into distinct parts, and  $\rho$  is an array of 1's or overlined parts less than or equal to  $k$ .

The process applied to  $A$  is the process established by Corteel and Lovejoy [8], and the process applied to  $B$  is a special case of the bijection discussed in Section 3. Thus the process above is reversible.

Case II:  $k < 0$ . By symmetry we can proceed in this case as we did with Case I. Because the proof is similar to the previous proof, we omit it.

**Example 3.** Let

$$\begin{pmatrix} \bar{6} & 5 & 4 & 4 & \bar{3} & 3 & \bar{2} & 2 & \bar{0} \\ 6 & \bar{5} & 5 & \bar{4} & 4 & 3 & \bar{1} & & \end{pmatrix}$$

be an F-partition. Rearrange the parts of the bottom row such that the overlined parts follow the unrestricted parts, and divide them into two arrays  $A$  and  $B$ .

$$\begin{pmatrix} \bar{6} & 5 & 4 & 4 & \bar{3} & 3 & \bar{2} & 2 & \bar{0} \\ 6 & \bar{5} & 5 & \bar{4} & 4 & 3 & \bar{1} & & \end{pmatrix} \longrightarrow B = \begin{pmatrix} \bar{6} & 5 & 4 & 4 & \bar{3} & 3 \\ 6 & 5 & 4 & 3 & & \end{pmatrix}, A = \begin{pmatrix} \bar{2} & 2 & \bar{0} \\ \bar{5} & \bar{4} & \bar{1} \end{pmatrix}$$

We build two partitions  $\alpha$  and  $\beta$  by inserting each column of  $A$ .

$$\begin{array}{ccc} \begin{pmatrix} \bar{2} & 2 & \bar{0} \\ \bar{5} & \bar{4} & \bar{1} \end{pmatrix} & \begin{matrix} \epsilon & \epsilon \end{matrix} & \\ \\ \begin{pmatrix} \bar{2} & 2 \\ \bar{5} & \bar{4} \end{pmatrix} & \begin{matrix} \circ \\ \bullet \end{matrix} & \begin{matrix} \epsilon \\ \epsilon \end{matrix} \\ \\ \begin{pmatrix} \bar{2} \\ \bar{5} \end{pmatrix} & \begin{matrix} \bullet & \bullet \\ \circ \\ \bullet \end{matrix} & \begin{matrix} (\bar{5}) \\ (\bar{5}) \end{matrix} \\ \\ \epsilon & \begin{matrix} \circ & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \circ & \bullet \\ \bullet & \bullet & \\ \bullet & & \\ \bullet & & \end{matrix} & \begin{matrix} (\bar{5}) \\ \beta \end{matrix} \\ \\ & \begin{matrix} \alpha \end{matrix} & \end{array}$$

We rearrange the top row of  $B$  and denote it by  $\sigma$ . We divide  $\sigma$  into  $\{\sigma\}$  and  $[\sigma]$ .

$$\sigma = (5 \ 4 \ 4 \ \bar{6} \ 3 \ \bar{3}) \longrightarrow \{\sigma\} = (0 \ 0 \ 0 \ \bar{2} \ 0 \ \bar{0}), \ [\sigma] = (5 \ 4 \ 4 \ 4 \ 3 \ 3)$$

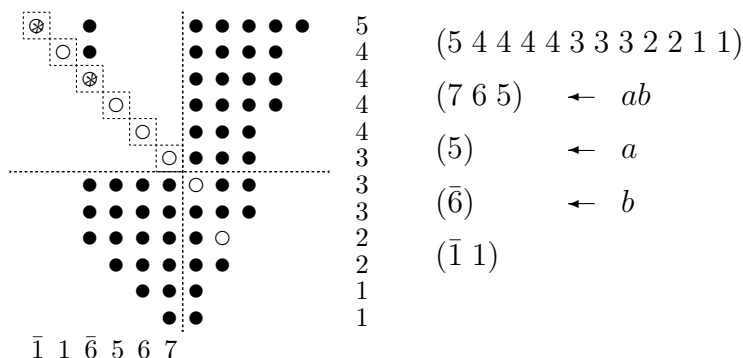
We increase each part of  $\{\sigma\}$  and take the last two parts for  $\rho$  and add each of the remaining parts of  $\{\sigma\}$  to the corresponding part of the bottom row of  $B$ .

$$\begin{pmatrix} 1 & 1 & 1 & \bar{3} \\ 6 & 5 & 4 & 3 \end{pmatrix}, \quad \rho = (1 \ \bar{1})$$

$$7 \ 6 \ 5 \ \bar{6}$$

We add the parts of  $[\sigma]$  to  $\alpha$ . Thus, we obtain

$$\alpha = (5\ 4\ 4\ 4\ 4\ 3\ 3\ 3\ 2\ 2\ 1\ 1), \quad \beta = (5), \quad \pi = (7\ 6\ 5), \quad \nu = (\bar{6}), \quad \rho = (1\ \bar{1}).$$



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### REFERENCES

[1] C. Adiga, B. C. Berndt, S. Bhargava, and G. N. Watson *Chapter 16 of Ramanujan's second notebook: Theta-functions and q-series*, Mem. Amer. Math. Soc., **53** No. 315 (1985).  
 [2] G. E. Andrews, *On Ramanujan's  ${}_1\psi_1(a; b; z)$* , Proc. Amer. Math. Soc., **22** (1969), 552–553.  
 [3] G. E. Andrews, *On a transformation of bilateral series with applications*, Proc. Amer. Math. Soc., **25** (1970), 554–558.  
 [4] G. E. Andrews, *Generalized Frobenius partitions*, Mem. Amer. Math. Soc., **49** No. 301 (1984).  
 [5] G. E. Andrews and R. Askey, *A simple proof of Ramanujan's  ${}_1\psi_1$* , Aequa. Math., **18** (1981), 97–100.  
 [6] R. Askey, *Ramanujan's extensions of the gamma and beta functions*, Amer. Math. Monthly, **87** (1980), 346–359.  
 [7] W. N. Bailey, *On the basic bilateral hypergeometric series  ${}_2\psi_2$* , Quateraly J. Math. (Oxford), 2<sup>nd</sup> series **1** (1950), 194–198.  
 [8] S. Corteel and J. Lovejoy, *Frobenius partitions and the combinatorics of Ramanujan's  ${}_1\psi_1$  summation*, J. Combin. Theory Ser. A, **97** (2002), 177–183.  
 [9] S. Corteel and J. Lovejoy, *Overpartitions*, Trans. Amer. Math. Soc., to appear.  
 [10] S. Corteel, *Particle seas and basic hypergeometric series*, Adv. Appl. Math., **31** (2003), 199–214.  
 [11] N. J. Fine, *Basic Hypergeometric Series and Applications*, Amer. Math. Soc., Providence, 1988.  
 [12] G. Frobenius, *Über die Charaktere der symmetrischen Gruppe*, Sitzber. Preuss. Akad. Berlin, 1900, 516–534.  
 [13] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990.  
 [14] W. Hahn, *Beiträge zur Theorie der Heineschen Reihen*, Math. Nachr., **2** (1949), 340–379.  
 [15] G. H. Hardy, *Ramanujan*, reprinted, the American Mathematical Society Chelsea, Providence, 1999.  
 [16] E. Heine, *Untersuchungen über die Reihe*

$$1 + \frac{(1 - q^\alpha)(1 - q^\beta)}{(1 - q)(1 - q^\gamma)}x + \frac{(1 - q^\alpha)(1 - q^{\alpha+1})(1 - q^\beta)(1 - q^{\beta+1})}{(1 - q)(1 - q^2)(1 - q^\gamma)(1 - q^{\gamma+1})}x^2 + \dots,$$

J. Reine Angew. Math., **34** (1847), 285–328.

[17] M. E. H. Ismail, *A simple proof of Ramanujan's  ${}_1\psi_1$  sum*, Proc. Amer. Math. Soc., **63** (1977), 185–186.

- [18] M. Jackson, *On Lerch's transcendent and the basic bilateral hypergeometric series  ${}_2\Psi_2$* , J. London Math. Soc., **25** (1950), 189–196.
- [19] J. T. Joichi and D. Stanton, *Bijective proofs of basic hypergeometric series identities*, Pacific J. Math., **127** (1987), 103–120.
- [20] K. Mimachi, *A proof of Ramanujan's identity by use of loop integral*, SIAM J. Math. Anal., **19** (1988), 1490–1493.
- [21] I. Pak, *Partition bijections*, preprint, (2002).
- [22] E. M. Wright, *An enumerative proof of an identity of Jacobi*, J. London Math. Soc., **40** (1965), 55–57.
- [23] A. J. Yee, *Combinatorial proofs of generating function identities for  $F$ -partitions*, J. Combin. Theory Ser. A, **102** (2003), 217–228.

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