INVERTIBLE AND ISOMETRIC COMPOSITION OPERATORS ON VECTOR-VALUED HARDY SPACES

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Abstract. Invertible and isometric composition operators acting on vector-valued Hardy space $H^2(E)$ are characterized.

1. Introduction

If $\phi$ is an analytic self-map of the unit disc $D$, the composition operator $C_\phi$ is defined by $C_\phi f = f \circ \phi$ for $f$ analytic in $D$. It is well known that every composition operator is bounded on scalar-valued Hardy Spaces $H^p$ as well as on other spaces of analytic functions. For detailed study of these operators on $H^p$ and other spaces of analytic functions, consult Schwartz [7], Shapiro and Taylor [8], Nordgren [5] and Cowen and MacCluer [1]. In this paper we study invertible and isometric composition operators on vector-valued Hardy space $H^2(E)$.

Let $(X, \|\cdot\|_X)$ and $(E, \langle\cdot,\cdot\rangle)$ denote a complex Banach space and a Hilbert space respectively. For $0 < p < \infty$, the vector-valued Hardy space $H^p(X)$ consists of functions $f : D \to X$ such that $x^*f$ is holomorphic in $D$ for every $x^* \in X^*$, the dual of $X$ and

$$
\lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|_X^p d\theta < \infty,
$$

where $D$ is the open unit disc in the complex plane $\mathbb{C}$ with boundary $\partial D$. For $1 \leq p < \infty$, $H^p(X)$ becomes a Banach space with norm $\|\cdot\|_p$ defined as

$$
\|f\|_p = \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|_X^p d\theta < \infty.
$$

When $X = \mathbb{C}$, we drop $X$ and write simply $H^p$ for $H^p(X)$ and $\|\cdot\|_p$ for $\|\cdot\|_p$.

Received July 15, 2002.
2000 Mathematics Subject Classification: Primary 47B33; Secondary 46E22.
Key words and phrases: vector-valued hardy spaces, harmonic majorant, inner function, invertible operator, and isometry.
In terms of harmonic majorants $H^p(X)$ consists of those holomorphic functions $f : D \to X$ for which $\|f(\cdot)\|_X^p$ has harmonic majorant and in this case (cf. [6, Theorem A, p.74])

$$\|f\|_P^p = h_f(0),$$

where $h_f$ is the least harmonic majorant of $\|f(\cdot)\|_X^p$.

A more detailed discussion of vector-valued analytic functions and Hardy spaces can be found in Hille and Philips [4], Rosenblum and Rovnyak [6], Hensgen [3] and a convenient reference for classical Hardy spaces is Duren [2].

We now prove the following lemma.

**Lemma 1.1.** Let $f \in H^p(X)$. Then

$$\|f(z)\|_X^p \leq \frac{2\|f\|_P^p}{1 - |z|} \text{ for every } z \in D.$$

**Proof.** Let $f \in H^p(X)$. Then $\|f(z)\|_X^p$ has the least harmonic majorant, say $h_f$. Therefore, by Harnack’s inequality,

$$\|f(z)\|_X^p \leq \frac{1 + |z|}{1 - |z|} h_f(0) \leq \frac{2}{1 - |z|} \|f\|_P^p.$$

A painless verification, using theorem C of Rosenblum and Rovnyak [6, p.76] and Harnack’s inequality shows that if $\phi : D \to D$ is analytic, then $C_\phi : H^p(X) \to H^p(X)$ is bounded and

$$\|C_\phi\|_P^p \leq \frac{1 + |\phi(0)|}{1 - |\phi(0)|}. \quad (1.1)$$

In case $p = 2$ and $X = E$, $H^p(E)$ becomes a Hilbert space with the inner product $\langle \langle \cdot, \cdot \rangle \rangle$ defined as

$$\langle \langle f, g \rangle \rangle = \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \langle f(re^{i\theta}), g(re^{i\theta}) \rangle d\theta.$$

Let $\{e_j : j \in J\}$ be an orthonormal basis for $E$, where $J$ is an index set of any size and let $N = \{0, 1, 2, \ldots\}$. For $(n, j) \in N \times J$, we define $e_{nj} : D \to E$ as

$$e_{nj}(z) = z^n e_j \text{ for every } z \in D.$$
Then \( \{ e_{nj} : (n, j) \in N \times J \} \) is an orthonormal basis for \( H^P(E) \). For \( z \in D \) and \( j \in J \), we define \( E^j_z : H^2(E) \rightarrow \mathbb{C} \) as
\[
E^j_z f = \langle f(z), e_{j} \rangle,
\]
for every \( f \in H^P(E) \). Then, by Lemma 1.1, \( E^j_z \) is a bounded linear functional on \( H^2(E) \). Hence by Riesz-representation theorem, there exists \( k^j_z \in H^2(E) \) such that
\[
E^j_z f = \langle \langle f, k^j_z \rangle \rangle,
\]
for every \( f \in H^2(E) \). We designate \( k^j_z \)'s as generalized reproducing kernels or simply kernel functions whenever there is no confusion. A straightforward calculation, using Parseval’s identity, shows that
\[
k^j_z(w) = \frac{e_j}{1 - \bar{w} z},
\]
for every \( w \in D \) and
\[
\|k^j_z\|_2^2 = \frac{1}{1 - |z|^2}.
\]
The invertibility of \( C_\phi \) on \( H^2(E) \) in terms of the invertibility of inducing map \( \phi \) is characterized in Section 2. We also present a necessary and sufficient condition for \( C_\phi \) to be an isometry.

2. Invertible and isometric composition operators

Schwartz [7] proved that \( C_\phi \) is invertible on \( H^P \) if and only if \( \phi \) is a conformal automorphism of the open unit disc. In the following theorem we generalize this criterion for invertibility of \( C_\phi \) to vector-valued Hardy space \( H^2(E) \) . The techniques applied to prove this result are different from those applied by Schwartz.

Theorem 2.1. \( C_\phi \) is invertible on \( H^2(E) \) if and only if \( \phi \) is invertible.

Before we prove this theorem we first note that if \( C_\phi \) is a composition operator on \( H^2(E) \), then \( C_\phi^* k^j_z = k^j_{\phi(z)} \), where \( C_\phi^* \) is the adjoint of \( C_\phi \). In fact,
\[
\langle \langle f, C_\phi^* k^j_z \rangle \rangle = \langle \langle C_\phi f, k^j_z \rangle \rangle = \langle f(\phi(z)), e_j \rangle = \langle \langle f, k^j_{\phi(z)} \rangle \rangle \text{ for every } f \in H^2(E).
\]
This implies that
\[
C_\phi^* k^j_z = k^j_{\phi(z)}.
\]
Proof of Theorem 2.1. If \( \phi \) is invertible, then \( C_\phi^{-1} = C_{\phi^{-1}} \).

Conversely, suppose \( C_\phi \) is invertible. Let \( \phi(z) = \phi(w) \) for some \( z, w \in D \). Then

\[
C_\phi^* k_z^j = k_{\phi(z)}^j = k_{\phi(w)}^j = C_\phi^* k_w^j.
\]

Since \( C_\phi \) and hence \( C_\phi^* \) is invertible, so \( k_z^j = k_{\phi(z)}^j \), which implies that \( z = w \). Hence \( \phi \) is univalent. Again, since \( C_\phi^* \) is invertible and so it is bounded below. Hence there exists \( \alpha > 0 \) such that

\[
|||C_\phi^* f|||^2 \geq \alpha |||f|||^2
\]
for every \( f \in H^2(E) \). In particular,

\[
|||C_\phi^* k_z^j|||^2 \geq \alpha |||k_w^j|||^2
\]
for every \( (z, j) \in D \times J \), and so by the above remark

\[
(2.1) \quad \frac{|||k_{\phi(z)}^j|||^2}{|||k_w^j|||^2} \geq \alpha \text{ for every } (z, j) \in D \times J.
\]

If \( \phi \) is not onto, then we can find \( w \in \partial \phi(D) \cap D \) and a sequence \( \{z_n\} \subset D \) such that

\[
\lim_n \phi(z_n) = w.
\]

By open mapping theorem \( |z_n| \to 1 \). Since

\[
|||k_{\phi(z_n)}^j|||^2 \to |||k_w^j|||^2, \quad |||k_{w}^j|||^2 < \infty,
\]
and \( |||k_{\phi(z_n)}^j|||^2 \to \infty \) as \( n \to \infty \), we conclude that

\[
\frac{|||k_{\phi(z_n)}^j|||^2}{|||k_{w}^j|||^2} \to 0 \text{ as } n \to \infty,
\]
a contradiction to (2.1). Hence \( \phi \) must be onto. This completes the proof of the theorem. \( \square \)

We next present a necessary and sufficient condition for \( C_\phi \) to be an isometry.

Theorem 2.2. \( C_\phi \) is an isometric on \( H^2(E) \) if and only if \( \phi \) is inner and \( \phi(0) = 0 \).

To prove this theorem we need the following lemmas.
Lemma 2.3. $||C_\phi||^2 \geq \frac{1}{1-|\phi(0)|^2}$.

Proof. Since $C_\phi^* k^j_z = k^j_{\phi(z)}$ for every $(z, j) \in D \times N$ and $|||k^j_o|||_2 = 1$, we have $\frac{1}{1-|\phi(0)|^2} = |||k^j_{\phi(o)}|||_2 = |||C_\phi^* k^j_o|||_2 \leq ||C_\phi||_2$. This completes the proof. \qed

Lemma 2.4. $||C_\phi|| = 1$ if and only if $\phi(0) = 0$.

Proof follows from the inequality (1.1) and Lemma 2.3.

Proof of theorem 2.2. Suppose $C_\phi$ is an isometry on $H^2(E)$. Then $|||C_\phi f|||_2 = |||f|||_2$ for every $f \in H^2(E)$. In particular, taking $f = e_{ij}$, we get, $\frac{1}{2\pi} \int_0^{2\pi} |\phi(e^{i\theta})|^2 d\theta = 1$ and so $\phi$ is inner. Also, since $C_\phi$ is an isometry $||C_\phi|| = 1$ and so by Lemma 2.4 $\phi(0) = 0$.

Conversely, suppose $\phi$ is inner and $\phi(0) = 0$. Then $\overline{\phi(e^{i\theta})} = [\phi(e^{i\theta})]^{-1}$ a.e. Further, if $f \in H^2(E)$, then $f(z) = \sum_{n=1}^{\infty} a_n z^n (a_n \in E)$ and $|||f|||_2 = \sum_{n=1}^{\infty} ||a_n||_2^2$ see ([6, section 1.15] and [4, chapter III]).

$$|||C_\phi f|||_2 = |||f_\phi|||_2^2$$

$$= \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \langle \sum_{m=0}^{\infty} a_m \phi^m(re^{i\theta}), \sum_{n=0}^{\infty} a_n \phi^n(re^{i\theta}) \rangle d\theta$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \langle a_m, a_n \rangle \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \phi^{m-n}(re^{i\theta}) d\theta$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \langle a_m, a_n \rangle \frac{1}{2\pi} \int_0^{2\pi} \phi^{m-n}(e^{i\theta}) d\theta$$

$$= \sum_{m,n \in N, m > n} \langle a_m, a_n \rangle \phi^{m-n}(0) + \sum_{m,n \in N, m < n} \langle a_m, a_n \rangle \phi^{m-n}(0)$$

$$+ \sum_{n=0}^{\infty} \langle a_n, a_n \rangle \frac{1}{2\pi} \int_0^{2\pi} \phi^n |(e^{i\theta})|^2 d\theta. (*)$$

Since $\phi^{m-n}(0) = 0$ for $m > n$, from (*) we have

$$|||C_\phi f|||_2^2 = \sum_{n=0}^{\infty} \langle a_n, a_n \rangle$$
\[
\sum_{n=0}^{\infty} ||a_n||_2^2 = ||f||_2^2
\]

Hence \( C_\phi \) is an isometry. \( \square \)

Acknowledgement. The authors wish to thank Professor R. K. Singh for his many suggestions and helpful comments. The authors are also thankful to the referee for pointing out typographical errors.

References


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