JENSEN TYPE QUADRATIC-QUADRATIC MAPPING IN BANACH SPACES

Choonkil Park, Seong-Ki Hong, and Myoung-Jung Kim

Abstract. Let $X, Y$ be vector spaces. It is shown that if an even mapping $f : X \to Y$ satisfies $f(0) = 0$ and

$$f \left( \frac{x+y}{2} + z \right) + f \left( \frac{x+y}{2} - z \right) + f \left( \frac{x-y}{2} + z \right)$$
$$+ f \left( \frac{x-y}{2} - z \right) = f(x) + f(y) + 4f(z)$$

for all $x, y, z \in X$, then the mapping $f : X \to Y$ is quadratic.

Furthermore, we prove the Cauchy–Rassias stability of the functional equation (0.1) in Banach spaces.

1. Introduction

In 1940, S. M. Ulam [19] raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?

Let $X$ and $Y$ be Banach spaces with norms $\| \cdot \|$ and $\| \cdot \|$, respectively. Hyers [5] showed that if $\epsilon > 0$ and $f : X \to Y$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \epsilon$$

for all $x \in X$.

Received March 14, 2005.
2000 Mathematics Subject Classification: 39B52.
Key words and phrases: Cauchy–Rassias stability, quadratic mapping, functional equation.
Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that
\[
\|f(x + y) - f(x) - f(y)\| \leq \epsilon(||x||^p + ||y||^p)
\]
for all $x, y \in X$. Th. M. Rassias [10] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that
\[
\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2p}||x||^p
\]

A square norm on an inner product space satisfies the important parallelogram equality
\[
\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.
\]
The functional equation
\[
f(x + y) + f(x - y) = 2f(x) + 2f(y)
\]
is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic function. A Hyers–Ulam stability problem for the quadratic functional equation was proved by Skof [17] for mappings $f : X \to Y$, where $X$ is a normed space and $Y$ is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain $X$ is replaced by an Abelian group. In [3], Czerwik proved the Cauchy–Rassias stability of the quadratic functional equation. Several functional equations have been investigated in [1] and [6]–[18].

In this paper, we solve the functional equation (0.1), and prove the Cauchy–Rassias stability of the functional equation (0.1) in Banach spaces.

2. Jensen type quadratic-quadratic mapping in Banach spaces

**Lemma 2.1.** Let $X$ and $Y$ be vector spaces. If an even mapping $f : X \to Y$ satisfies $f(0) = 0$ and
\[
f\left(\frac{x + y}{2} + z\right) + f\left(\frac{x + y}{2} - z\right) + f\left(\frac{x - y}{2} + z\right) + f\left(\frac{x - y}{2} - z\right)
\]
\[
= f(x) + f(y) + 4f(z)
\]
for all $x, y, z \in X$, then the mapping $f : X \to Y$ is quadratic.
Proof. Letting $x = y$ in (2.1), we get
\[
f(x + z) + f(x - z) + f(z) + f(-z) = 2f(x) + 4f(z)
\]
for all $x, z \in X$. Since $f(-z) = f(z)$,
\[
f(x + z) + f(x - z) = 2f(x) + 2f(z)
\]
for all $x, z \in X$. So the even mapping $f : X \to Y$ is quadratic. □

The mapping $f : X \to Y$ given in the statement of Lemma 2.1 is called a Jensen type quadratic-quadratic mapping. Putting $z = 0$ in (2.1), we get the Jensen type quadratic mapping
\[
2f(x + y) + 4f(z)
\]
for all $x, y, z \in X$. From now on, assume that $X$ is a normed vector space with norm $\| \cdot \|$ and that $Y$ is a Banach space with norm $\| \cdot \|$.

For a given mapping $f : X \to Y$, we define
\[
Df(x, y, z) := f\left(\frac{x + y}{2} + z\right) + f\left(\frac{x + y}{2} - z\right) + f\left(\frac{x - y}{2} + z\right)
\]
\[
+ f\left(\frac{x - y}{2} - z\right) - f(x) - f(y) - 4f(z)
\]
for all $x, y, z \in X$.

**THEOREM 2.2.** Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^3 \to [0, \infty)$ such that
\[
\tilde{\varphi}(x, y, z) := \sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty,
\]
\[
\|Df(x, y, z)\| \leq \varphi(x, y, z)
\]
for all $x, y, z \in X$. Then there exists a unique Jensen type quadratic-quadratic mapping $Q : X \to Y$ such that
\[
\|f(x) - Q(x)\| \leq \frac{1}{4}\tilde{\varphi}(x, x, x)
\]
for all $x \in X$. 
Proof. Letting $x = y = z$ in (2.3), we get
\begin{equation}
\|f(2x) - 4f(x)\| \leq \varphi(x, x, x)
\end{equation}
for all $x \in X$. So
\begin{equation}
\|f(x) - 4f(\frac{x}{2})\| \leq \varphi \left( \frac{x}{2}, \frac{x}{2}, \frac{x}{2} \right)
\end{equation}
for all $x \in X$. Hence
\begin{equation}
\left\| 4^lf(\frac{x}{2^l}) - 4^mf(\frac{x}{2^m}) \right\| \leq \sum_{j=l+1}^{m} 4^{j-1} \varphi \left( \frac{x}{2^j}, \frac{x}{2^j}, \frac{x}{2^j} \right)
\end{equation}
for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.2) and (2.6) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by
\[ Q(x) := \lim_{n \to \infty} 4^n f \left( \frac{x}{2^n} \right) \]
for all $x \in X$.

By (2.3) and (2.2),
\begin{align*}
\|DQ(x, y, z)\| &= \lim_{n \to \infty} 4^n \left\| Df \left( \frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n} \right) \right\| \\
&\leq \lim_{n \to \infty} 4^n \varphi \left( \frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n} \right) = 0
\end{align*}
for all $x, y, z \in X$. So $DQ(x, y, z) = 0$. By Lemma 2.1, the mapping $Q : X \to Y$ is a Jensen type quadratic-quadratic mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get the inequality (2.4).

Now, let $Q' : X \to Y$ be another Jensen type quadratic-quadratic mapping satisfying (2.4). Then we have
\begin{align*}
\|Q(x) - Q'(x)\| &= 4^n \left\| Q(\frac{x}{2^n}) - Q'(\frac{x}{2^n}) \right\| \\
&\leq 4^n \left( \left\| Q(\frac{x}{2^n}) - f(\frac{x}{2^n}) \right\| + \|Q'(\frac{x}{2^n}) - f(\frac{x}{2^n})\| \right) \\
&\leq 2 \cdot 4^n \frac{\varphi}{4} \left( \frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n} \right),
\end{align*}
which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = Q'(x)$ for all $x \in X$. This proves the uniqueness of $Q$. \qed
Corollary 2.3. Let $p$ and $\theta$ be positive real numbers with $p > 2$, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and
\[
\|Df(x, y, z)\| \leq \theta(||x||^p + ||y||^p + ||z||^p)
\]
for all $x, y, z \in X$. Then there exists a unique Jensen type quadratic-quadratic mapping $Q : X \to Y$ such that
\[
\|f(x) - Q(x)\| \leq \frac{3\theta}{2p - 4}||x||^p
\]
for all $x \in X$.

Proof. Define $\varphi(x, y, z) = \theta(||x||^p + ||y||^p + ||z||^p)$, and apply Theorem 2.2. \qed

Theorem 2.4. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^3 \to [0, \infty)$ satisfying (2.3) such that
\[
(2.7) \quad \tilde{\varphi}(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{4^j} \varphi(2^j x, 2^j y, 2^j z) < \infty
\]
for all $x, y, z \in X$. Then there exists a unique Jensen type quadratic-quadratic mapping $Q : X \to Y$ such that
\[
(2.8) \quad \|f(x) - Q(x)\| \leq \frac{1}{4} \tilde{\varphi}(x, x, x)
\]
for all $x \in X$.

Proof. It follows from (2.5) that
\[
\|f(x) - \frac{1}{4} f(2x)\| \leq \frac{1}{4} \varphi(x, x, x)
\]
for all $x \in X$. Hence
\[
(2.9) \quad \|\frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x)\| \leq \sum_{j=l}^{m-1} \frac{1}{4^{j+1}} \varphi(2^j x, 2^j x, 2^j x)
\]
for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.7) and (2.9) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence.
for all $x \in X$. Since $Y$ is complete, the sequence $\{1/4^n f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n \to \infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$.

By (2.7) and (2.3),

$$\|DQ(x, y, z)\| = \lim_{n \to \infty} \frac{1}{4^n} \|Df(2^n x, 2^n y, 2^n z)\| \leq \lim_{n \to \infty} \frac{1}{4^n} \varphi(2^n x, 2^n y, 2^n z) = 0$$

for all $x, y, z \in X$. So $DQ(x, y, z) = 0$. By Lemma 2.1, the mapping $Q : X \to Y$ is a Jensen type quadratic-quadratic mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.9), we get the inequality (2.8).

The rest of the proof is similar to the proof of Theorem 2.2. □

**Corollary 2.5.** Let $p$ and $\theta$ be positive real numbers with $p < 2$, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and

$$\|Df(x, y, z)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$. Then there exists a unique Jensen type quadratic-quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{3\theta}{4 - 2p} \|x\|^p$$

for all $x \in X$.

**Proof.** Define $\varphi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$, and apply Theorem 2.4. □

**References**


Jensen type quadratic-quadratic mapping


Choonkil Park, Department of Mathematics, Hanyang University, Seoul 133-791, Korea
E-mail: baak@hanyang.ac.kr

Seong-Ki Hong and Myoung-Jung Kim, Department of Mathematics, Chungnam National University, Daejeon 305-764, Korea
E-mail: mrhongsk@hanmail.net
mjkim@math.cnu.ac.kr