SASAKIAN MANIFOLDS WITH QUASI-CONFORMAL CURVATURE TENSOR

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Abstract. The object of the paper is to study a Sasakian manifold with quasi-conformal curvature tensor.

1. Introduction

The notion of the quasi-conformal curvature tensor was given by Yano and Sawaki [11]. According to them a quasi-conformal curvature tensor \( \tilde{C} \) is defined by

\[
\tilde{C}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)Q_X - g(X,Z)Q_Y]
- \frac{r}{n} \left[ \frac{a}{n - 1} + 2b \right] [g(Y,Z)X - g(X,Z)Y],
\]

where \( a \) and \( b \) are constants and \( R, S, Q, \) and \( r \) are the Riemannian curvature tensor of type (1,3), the Ricci tensor of type (0,2), the Ricci operator defined by \( g(QX, Y) = S(X, Y) \) and the scalar curvature of the manifold respectively. If \( a = 1 \) and \( b = -\frac{1}{n-2} \), then (1.1) takes the form

\[
\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{1}{n-2} [S(Y,Z)X - S(X,Z)Y + g(Y,Z)Q_X
- g(X,Z)Q_Y] + \frac{r}{(n-1)(n-2)} [g(Y,Z)X - g(X,Z)Y]
= C(X,Y)Z,
\]

where \( C \) is the conformal curvature tensor [4]. Thus the conformal curvature tensor \( C \) is a particular case of the tensor \( \tilde{C} \). For this reason \( \tilde{C} \) is called the quasi-conformal curvature tensor. A manifold \( (M^n, g)(n > 3) \) shall be called quasi-conformally flat if the quasi-conformal curvature tensor \( \tilde{C} = 0 \). It is known [1] that the quasi-conformally flat manifold is either conformally flat if \( a \neq 0 \) or, Einstein if \( a = 0 \) and \( b \neq 0 \). Since, they give no restrictions for manifolds if \( a = 0 \) and \( b = 0 \), it is essential for us to consider the case of \( a \neq 0 \) or \( b \neq 0 \).

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An almost contact metric manifold is said to be an \(\eta\)-Einstein manifold if the Ricci tensor \(S\) satisfies the condition
\[
S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),
\]
where \(a, b\) are certain scalars. It is known \([10]\) that in a Sasakian manifold \(a, b\) are constants. A Riemannian or a semi-Riemannian manifold is said to be semi-symmetric \([8]\) if \(R(X, Y) \circ R = 0\), where \(R\) is the Riemannian curvature tensor and \(R(X, Y)\) is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors \(X, Y\). If a Riemannian manifold satisfies \(R(X, Y) \cdot \tilde{C} = 0\), where \(\tilde{C}\) is the quasi-conformal curvature tensor, then the manifold is said to be quasi-conformally semi-symmetric manifold.

It is known \([5]\) that a conformally flat Sasakian manifold is of constant curvature and a Weyl semi-symmetric Sasakian manifold is locally isometric with the unit sphere \(S^n(1)\) \([3]\). In the present paper we have studied quasi-conformally flat and quasi-conformally semi-symmetric Sasakian manifolds. At first we prove that a Sasakian manifold is quasi-conformally flat if and only if it is locally isometric with the unit sphere \(S^n(1)\). Also it is proved that a compact orientable quasi-conformally flat Sasakian manifold can not admit a non-isometric conformal transformation. Finally, we have shown that a Sasakian manifold is quasi-conformally flat if and only if it is quasi-conformally semi-symmetric.

2. Preliminaries

Let \(S\) and \(r\) denote respectively the Ricci tensor of type \((0,2)\) and the scalar curvature in a Sasakian manifold \((M^n, g)\). It is known that in a Sasakian manifold \(M^n\), the following relations hold \([6], [2], [7]\):

\[
\begin{align*}
(2.1) & \quad \phi(\xi) = 0 \\
(2.2) & \quad \eta(\xi) = 1 \\
(2.3) & \quad \phi^2 X = -X + \eta(X)\xi \\
(2.4) & \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \\
(2.5) & \quad g(\xi, X) = \eta(X) \\
(2.6) & \quad \nabla X \xi = -\phi X \\
(2.7) & \quad S(X, \xi) = (n-1)\eta(X) \\
(2.8) & \quad g(R(\xi, X)Y, \xi) = g(X, Y) - \eta(X)\eta(Y) \\
(2.9) & \quad R(\xi, X)\xi = -X + \eta(X)\xi \\
(2.10) & \quad g(R(X, Y)\xi, Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X) \\
(2.11) & \quad (\nabla_X \phi)(Y) = R(\xi, X)Y
\end{align*}
\]

for any vector fields \(X, Y\).

The above results will be used in the next section.
3. $\eta$-Einstein Sasakian manifold

Let $l^2$ be the square of the length of the Ricci tensor, then

\begin{equation}
(3.1) \quad l^2 = \sum_{i=1}^{n} S(Qe_i, e_i),
\end{equation}

where $Q$ is the symmetric endomorphism of the tangent space at a point corresponding to the Ricci tensor $S$ and $\{e_i\}$, $i = 1, 2, \ldots, n$, is an orthonormal basis of the tangent space at any point. Now putting $X = Y = \{e_i\}$ in (1.2), and taking summation over $i$, $1 \leq i \leq n$, we get

\begin{equation}
(3.2) \quad r = na + b,
\end{equation}

where $r$ is the scalar curvature. Again from (1.2) we obtain

\begin{equation}
(3.3) \quad S(\xi, \xi) = a + b.
\end{equation}

Now we get from (1.2) with the help of (3.1), (3.2) and (3.3)

\begin{equation}
(3.4) \quad l^2 = (n - 1)a^2 + (a + b)^2.
\end{equation}

Since the scalars $a$ and $b$ are constants of an $\eta$-Einstein Sasakian manifold, it follows from (3.2) that $r$ is constant and so also is the length of the Ricci tensor. Next we suppose that the manifold under consideration admits a non-isometric conformal motion generated by a vector field $X$. Since $l^2$ is constant, it follows that

\begin{equation}
(3.5) \quad L_Xl^2 = 0,
\end{equation}

where $L_X$ denotes Lie-differentiation with respect to $X$. Now it is known [9] that if a compact Reimannian manifold $M^n(n > 2)$ with constant scalar curvature admits an infinitesimal nonisometric conformal transformation $X$ such that $L_Xl^2 = 0$, then $M$ is isometric to a sphere. But a sphere is an Einstein manifold. Hence we can state the following:

**Theorem 3.1.** A compact orientable $\eta$-Einstein Sasakian manifold does not admit a nonisometric conformal transformation.

4. Quasi-conformally flat Sasakian manifold

If the manifold under consideration is quasi-conformally flat, then we have from (1.1)

\begin{equation}
(4.1) \quad 'R(X, Y, Z, W) = \frac{b}{a}[S(X, Z)g(Y, W) - S(Y, Z)g(X, W) + S(Y, W)g(X, Z) - S(X, W)g(Y, Z)] + \frac{r}{na} \left( \frac{a}{n - 1} + 2b \right) [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)],
\end{equation}

where $a$ and $b$ are constants and $'R(X, Y, Z, W) = g(R(X, Y)Z, W)$. 
Now putting $Z = \xi$ in (4.1) and using (2.2), (2.5), (2.7) and (2.10) we get
\begin{align*}
g(X, W)\eta(Y) - g(Y, W)\eta(X) \\
= \frac{b}{a}[(n-1)g(Y, W)\eta(X) - (n-1)g(X, W)\eta(Y)] \\
+ S(Y, W)\eta(X) - S(X, W)\eta(Y)] \\
+ \frac{r}{na} [\frac{a}{n-1}] \\
+ 2b [g(X, W)\eta(Y) - g(Y, W)\eta(X)].
\end{align*}
\tag{4.2}
Again putting $X = \xi$ in (4.2) and using (2.2), (2.5) and (2.7) it follows that
\begin{equation}
S(Y, W) = Ag(Y, W) + B\eta(Y)\eta(W),
\end{equation}
where
\begin{equation}
A = [-n-1 + \frac{r}{nb}(\frac{a}{n-1} + 2b) - \frac{a}{b}]
\end{equation}
and
\begin{equation}
B = [2(n-1) - \frac{r}{nb}(\frac{a}{n-1} + 2b) + \frac{a}{b}].
\end{equation}
Here $A + B = (n-1)$. This leads to the following theorem:

**Theorem 4.1.** A quasi-conformally flat Sasakian manifold is an $\eta$-Einstein manifold.

Now from Theorem 3.1 we can state the following:

**Corollary 4.1.** A compact orientable quasi-conformally flat Sasakian manifold can not admit a nonisometric conformal transformation.

Putting $Y = W = \{e_i\}$ in (4.3) and taking summation over $i$, $1 \leq i \leq n$, we get
\begin{equation}
r = nA + B.
\end{equation}
Now with the help of (4.4) and (4.5) the equation (4.6) gives
\begin{equation}
[(n-2) + \frac{a}{b}]\frac{r}{n} + (1-n) = 0.
\end{equation}
Hence either
\begin{equation}
b = \frac{a}{2-n},
\end{equation}
or,
\begin{equation}
r = n(n-1).
\end{equation}

If $b = \frac{a}{2-n}$ then putting it into (1.1) we get
\begin{equation}
\tilde{C}(X, Y)Z = aC(X, Y)Z,
\end{equation}
where $C(X, Y)Z$ denotes the Weyl conformal curvature tensor. So the quasi-conformally flatness and conformally flatness are equivalent in this case. A conformally flat Sasakian manifold $(M^n, g)(n \geq 5)$ is of constant curvature. But a manifold of constant curvature is conformally flat. Hence a Sasakian
manifold is conformally flat if and only if it is locally isometric with a unit sphere \( S^n(1) \). So in this case \( M^n \) is locally isometric to the unit sphere.

If \( r = n(n - 1) \), then from (4.3), (4.4) and (4.5) we obtain

\[
S(Y, W) = (n - 1)g(Y, W).
\]

This implies that \( M^n \) is an Einstein manifold. So putting (4.8), (4.9) and (4.11) into (4.1) we obtain

\[
R(X, Y, Z, W) = g(X, W)g(Y, Z) - g(X, Z)g(Y, W).
\]

Then \( M^n \) is of constant curvature +1. Hence it is locally isometric with the unit sphere \( S^n(1) \). If \( M^n \) is locally isometric to the unit sphere \( S^n(1) \) then it is easy to see that \( M^n \) is quasi-conformally flat. This leads to the following theorem:

**Theorem 4.2.** Let \((M^n, g)(n \geq 5)\) be a Sasakian manifold. Then \( M^n \) is quasi-conformally flat if and only if \( M^n \) is locally isometric to the unit sphere \( S^n(1) \).

5. Sasakian manifolds satisfying \( R(X, Y) \cdot \tilde{C} = 0 \)

In this section we consider a Sasakian manifold \( M^n \) satisfying the condition

\[
R(X, Y) \cdot \tilde{C} = 0.
\]

Then we obtain from (1.1) by using (2.5), (2.7) and (2.10)

\[
\eta(\tilde{C}(X, Y)Z) = [a + b(n - 1) - \frac{r}{n}\left(\frac{a}{n - 1} + 2b\right)]g(Y, Z)\eta(X)
- g(X, Z)\eta(Y)] + b[S(Y, Z)\eta(X) - S(X, Z)\eta(Y)].
\]

For \( Z = \xi \), we get from (5.2)

\[
\eta(\tilde{C}(X, Y)\xi) = 0.
\]

Again putting \( X = \xi \) in (5.2) we get

\[
\eta(\tilde{C}(\xi, Y)Z) = [a + b(n - 1) - \frac{r}{n}\left(\frac{a}{n - 1} + 2b\right)]g(Y, Z)
- \eta(Y)\eta(Z)] + b[S(Y, Z) - (n - 1)\eta(Y)\eta(Z)].
\]

In virtue of (5.1) we get

\[
R(X, Y)\tilde{C}(U, V)W - \tilde{C}(R(X, Y)U, V)W
- \tilde{C}(U, R(X, Y)V)W - \tilde{C}(U, V)R(X, Y)W = 0,
\]

which implies that

\[
'\tilde{C}(U, V, W, Y) - \eta(Y)\eta(\tilde{C}(U, V)W) + \eta(U)\eta(\tilde{C}(Y, V)W)
+ \eta(V)\eta(\tilde{C}(U, Y)W) + \eta(W)\eta(\tilde{C}(U, V)Y) - g(Y, U)\eta(\tilde{C}(\xi, V)W)
- g(Y, V)\eta(\tilde{C}(U, \xi)W) - g(Y, W)\eta(\tilde{C}(U, V)\xi) = 0,
\]

where \( '\tilde{C}(U, V, W, Y) = g(\tilde{C}(U, V)W, Y) \).
Putting $U = Y$ in (5.6) and with the help of (5.2) and (5.3) we get
\begin{equation}
\begin{split}
\tilde{\mathcal{C}}(U, V, W, U) + \eta(W)\eta(\tilde{\mathcal{C}}(U, V)U) \\
- g(U, U)\eta(\tilde{\mathcal{C}}(\xi, V)W) - g(U, V)\eta(\tilde{\mathcal{C}}(U, \xi)W) = 0.
\end{split}
\end{equation}
Now putting $U = \{e_i\}$, where $\{e_i\}, i = 1, 2, \ldots, n$, be an orthonormal basis of the tangent space at each point of the manifold, in (5.7) and taking the summation over $i$, $1 \leq i \leq n$, and using (5.2), (5.4) we get
\begin{equation}
S(V, W) = \lambda g(V, W) + \mu \eta(V)\eta(W),
\end{equation}
where
\begin{equation}
\lambda = \frac{-br + (n - 1)^2b + (n - 1)a}{a - b}
\end{equation}
and
\begin{equation}
\mu = \frac{b[r - n(n - 1)]}{a - b}.
\end{equation}
Hence (5.8) leads to the following theorem:

**Theorem 5.1.** A quasi-conformally semi-symmetric Sasakian manifold is an $\eta$-Einstein manifold.

Now contracting (5.8) we get
\begin{equation}
r = n\lambda + \mu.
\end{equation}
By (5.9) and (5.10) the equation (5.11) gives
\begin{equation}
[a + (n - 2)b][r - n(n - 1)] = 0.
\end{equation}
Therefore, either
\begin{equation}
b = \frac{a}{2 - n} \quad \text{or} \quad r = n(n - 1).
\end{equation}
From (5.9) and (5.12) we obtain
\begin{equation}
\lambda = (n - 1).
\end{equation}
By (5.10) and (5.12) we get
\begin{equation}
\mu = 0.
\end{equation}
So, from (5.8), (5.13) and (5.14) we have
\begin{equation}
S(V, W) = (n - 1)g(V, W).
\end{equation}
Therefore, $M^n$ is an Einstein manifold. Now with the help of (5.12) and (5.15) the equations (5.2) and (5.4) imply that
\begin{equation}
\eta(\tilde{\mathcal{C}}(U, V)W) = 0
\end{equation}
and
\begin{equation}
\eta(\tilde{\mathcal{C}}(\xi, U)V) = 0,
\end{equation}
\begin{equation}
\eta(\tilde{\mathcal{C}}(U, V)W) = 0
\end{equation}
and
\begin{equation}
\eta(\tilde{\mathcal{C}}(\xi, U)V) = 0,
\end{equation}
where
respectively. So using (5.16), (5.17) and (5.3) into the equation (5.6) we get (5.18)
\[ \tilde{\mathcal{C}}(U, V, W, Y) = 0. \]
Therefore, \( M^n \) is quasi-conformally flat. Then it is trivially quasi-conformally semi-symmetric. So we have the following result:

**Theorem 5.2.** Let \((M^n, g)(n > 3)\) be a Sasakian manifold. Then \( M^n \) is quasi-conformally flat if and only if it is quasi-conformally semi-symmetric.

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