THE ZERO-DIVISOR GRAPH UNDER GROUP ACTIONS IN A NONCOMMUTATIVE RING

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Abstract. Let $R$ be a ring with identity, $X$ the set of all nonzero, nonunits of $R$ and $G$ the group of all units of $R$. First, we investigate some connected conditions of the zero-divisor graph $\Gamma(R)$ of a noncommutative ring $R$ as follows: (1) if $\Gamma(R)$ has no sources and no sinks, then $\Gamma(R)$ is connected and diameter of $\Gamma(R)$, denoted by $\text{diam}(\Gamma(R))$ (resp. girth of $\Gamma(R)$, denoted by $g(\Gamma(R))$) is equal to or less than 3; (2) if $X$ is a union of finite number of orbits under the left (resp. right) regular action on $X$ by $G$, then $\Gamma(R)$ is connected and $\text{diam}(\Gamma(R))$ (resp. $g(\Gamma(R))$) is equal to or less than 3, in addition, if $R$ is local, then there is a vertex of $\Gamma(R)$ which is adjacent to every other vertices in $\Gamma(R)$; (3) if $R$ is unit-regular, then $\Gamma(R)$ is connected and $\text{diam}(\Gamma(R))$ (resp. $g(\Gamma(R))$) is equal to or less than 3. Next, we investigate the graph automorphisms group of $\Gamma(\text{Mat}_2(\mathbb{Z}_p))$ where $\text{Mat}_2(\mathbb{Z}_p)$ is the ring of 2 by 2 matrices over the galois field $\mathbb{Z}_p$ ($p$ is any prime).

1. Introduction and basic definitions

The zero-divisor graph of a commutative ring has been studied extensitively by Akbari, Anderson, Frazier, Lauve, Livingston, and Maimani in [1, 2, 3] since its concept had been introduced by Beck in [4]. Recently, the zero-divisor graph of a noncommutative ring (resp. a semigroup) has also been studied by Redmond and Wu (resp. F. DeMeyer and L. DeMeyer) in [12, 13, 14] (resp. [6]). The zero-divisor graph is very useful to find the algebraic structures and properties of rings. In this paper, the zero-divisor graph of a noncommutative ring is also studied by considering some group actions.

Throughout this paper all rings are assumed to be rings with identity. For a ring $R$, let $Z_0(R)$ (resp. $Z_r(R)$) be the set of all left (resp. right) zero-divisors of $R$, $Z(R) = Z_0(R) \cup Z_r(R)$ and $\Gamma(R)$ be the zero-divisor graph of $R$ consisting of all vertices in $Z(R)^* = Z(R) \setminus \{0\}$, the set of all nonzero left or right zero-divisors of $R$, and edges $x \rightarrow y$, which means that $xy = 0$ for $x, y \in Z(R)^*$. If there exist vertices $x_0, \ldots, x_n \in Z(R)^*$ such that $P$:
by $x \in X$ of a finite regular action on $X$, it has been shown that if $P$ is a path from $x_0$ to $x_n$ of length $n$. We will denote $d(x, y)$ by the length of the shortest path from $x$ to $y$, otherwise, $d(x, y) = \infty$. Recall that $\Gamma(R)$ is connected if for all distinct vertices $x, y \in Z(R)^*$ there exists a path from $x$ to $y$. The diameter of $\Gamma(R)$ (denoted by $\text{diam}(\Gamma(R))$) is defined by the supremum of $d(x, y)$ for all distinct vertices $x$ and $y$ in $\Gamma(R)$. In particular, if $x = y$ and $d(x, x) = k$, then the path is called the cycle of length $k$. Usually vertices of a path may be considered to be distinct, however in a cycle, the initial and the final vertices are the same. If $\Gamma(R)$ contains a cycle, then the girth of $\Gamma(R)$ (denoted by $g(\Gamma(R))$) is defined by the length of the shortest cycle in $\Gamma(R)$, otherwise, $g(\Gamma(R)) = \infty$. In [7, Proposition 1.3.2], if $\Gamma(R)$ contains a cycle, then $1 + 2\text{diam}(\Gamma(R)) \geq g(\Gamma(R))$.

We say that $\Gamma(R)$ is complete if $xy = 0$ for any distinct vertices $x, y$ in $\Gamma(R)$.

For a ring $R$, let $X(R)$ (simply, denoted by $X$) be the set of all nonzero, nonunits of $R$, $G(R)$ (simply, denoted by $G$) be the group of all units of $R$ and $J(R)$ (simply, denoted by $J$) be the Jacobson radical of $R$. In this paper, we will consider some group actions on $X$ by $G$ given by $(g, x) \mapsto gx$ (resp. $(g, x) \mapsto xg^{-1}$) from $G \times X$ to $X$, called the left (resp. right) regular action. If $\phi : G \times X \rightarrow X$ is the left (resp. right) regular action, then for each $x \in X$, we define the orbit of $x$ by $\phi(x) = \{\phi(g, x) = gx : \forall g \in G\}$ (resp. $o_r(x) = \{\phi(g, x) = xg^{-1} : \forall g \in G\}$). Recall that $G$ is transitive on $X$ (or $G$ acts transitively on $X$) under the regular action on $X$ by $G$ if there is an $x \in X$ with $o_l(x) = X$ (resp. $o_r(x) = X$) and the left (resp. right) regular action on $X$ by $G$ is trivial if $o_l(x) = \{x\}$ (resp. $o_r(x) = \{x\}$) for all $x \in X$. In [8], it has been shown that if $X$ is a union of a finite number of orbits under the left regular action on $X$ by $G$, then $x^{n+1} = 0$ for all $x \in J$ and $X$ is the set of all nonzero right zero-divisors of $R$. Similarly, it is also shown that if $X$ is a union of a finite number of orbits under the right regular action on $X$ by $G$, then $x^{n+1} = 0$ for all $x \in J$ and $X$ is the set of all nonzero left zero-divisors of $R$.

Recall that for all $x \in X$ the set $\text{ann}_l(x) = \{y \in X : yx = 0\}$ (resp. $\text{ann}_r(x) = \{z \in X : zx = 0\}$) is called a left (resp. right) annihilator of $x$. Let $\text{ann}_l^*(x) = \text{ann}_l(x) \setminus \{0\}$ (resp. $\text{ann}_r^*(x) = \text{ann}_r(x) \setminus \{0\}$). Given a zero-divisor graph $\Gamma(R)$ and a vertex $x \in Z(R)^*$, the indegree (resp. outdegree) of $x$ (denoted by $\text{indegree}(x)$ (resp. $\text{outdegree}(x)$) is the number of edges arriving (resp. leaving) at $x$. That is, $\text{indegree}(x) = |\text{ann}_l^*(x)|$ (resp. $\text{outdegree}(x) = |\text{ann}_r^*(x)|$). A vertex of indegree 0 (resp. outdegree 0) is called a source (resp. sink).

In Section 2, some connected conditions of the zero-divisor graph of a non-commutative ring $R$ are investigated as follows: (1) if $\Gamma(R)$ has no sources and no sinks, then $\Gamma(R)$ is connected and $\text{diam}(\Gamma(R))$ (resp. $g(\Gamma(R))$) is equal to or less than 3; (2) if $X$ is a union of finite number of orbits under the left (resp. right) regular action on $X$ by $G$, then $\Gamma(R)$ is connected and $\text{diam}(\Gamma(R))$ (resp. $g(\Gamma(R))$) is equal to or less than 3, in addition, if $R$ is a local ring, then there exists a vertex of $\Gamma(R)$ which is adjacent to every other vertices in $\Gamma(R)$;
(4) if \( R \) is a unit-regular ring, then \( \Gamma(R) \) is connected and \( \text{diam}(\Gamma(R)) \) (resp. \( g(\Gamma(R)) \)) is equal to or less than 3.

In [3], Anderson and Livingston have shown that distinct ring automorphisms of a finite ring \( R \) which is not a field induce distinct graph automorphisms of \( \Gamma(R) \) and determined \( \text{Aut}(\Gamma(R)) \), the graph automorphisms group of \( \Gamma(R) \). In particular, they have computed \( \text{Aut}(\Gamma(\mathbb{Z}_n)) \).

In Section 3, when \( R = \text{Mat}_2(\mathbb{Z}_p) \), the ring of 2 by 2 matrices over the Galois field \( \mathbb{Z}_p \) (\( p \) is any prime), we will show that \( \text{Aut}(\Gamma(R)) \) is isomorphic to the group \( S_{p+1} \), the symmetric group of degree \( p+1 \) by investigating that (1) the number of orbits under the left (resp. right) regular action on \( X \) by \( G \) is \( p+1 \); (2) the number of nonzero nilpotents in \( R \) is \( p^2 - 1 \); (3) \( \text{Aut}(\Gamma(R)) \neq \{1\} \); (4) under the left (resp. right) regular action on \( X \) by \( G \), \( o \ell(a) \cap N(p) = o_r(a) \cap N(p) = o_r(a) \cap o_r(a) \) for all \( a \in N(p) \) where \( N(p) \) is the set of all nonzero nilpotents in \( R \).

2. Connected zero-divisor graph under the left (resp. right) regular action

For a subset \( S \) of \( Z(R)^* \), we will denote the subgraph of \( \Gamma(R) \) with vertices in \( S \) by \( \Gamma_S(R) \).

**Proposition 2.1.** Let \( R \) be a ring. If the left (or right) regular action of \( G \) on \( X \) is transitive, then \( \Gamma_X(R) \) is complete.

**Proof.** Since the left regular action of \( G \) on \( X \) is transitive, \( R \) is a local ring and \( J^2 = 0 \) by [8, Corollary 2.4], and so \( Z(R)^* = X \) and \( \Gamma_X(R) \) is complete. If the right regular action of \( G \) on \( X \) is transitive, then \( Z(R)^* = X \) and \( \Gamma_X(R) \) is also complete by the similar argument. \( \square \)

**Remark 1.** In Proposition 2.1, we see that if the left (resp. right) regular action on \( X \) by \( G \) is transitive, then \( x^2 = 0 \), i.e., \( x \) is a nilpotent element of nilpotency 2 for all \( x \in X \).

**Theorem 2.2.** Let \( R \) be a ring. If \( \Gamma(R) \) has no sources and no sinks, then \( \Gamma(R) \) is connected and \( \text{diam}(\Gamma(R)) \) (resp. \( g(\Gamma(R)) \)) is equal to or less than 3.

**Proof.** Let \( x, y \in Z(R)^*(x \neq y) \) be arbitrary. Since \( \Gamma(R) \) has no sources and no sinks, i.e., \( \text{ann}^*_r(x) \neq \emptyset \) (resp. \( \text{ann}^*_l(x) \neq \emptyset \)), there exists an element \( a \in X \) (resp. \( b \in X \)) such that \( xa = 0 \) (resp. \( by = 0 \)). If \( ab = 0 \), then \( x \rightarrow a \rightarrow b \rightarrow y \) is a path of length 3. If \( ab \neq 0 \), then \( x \rightarrow ab \rightarrow y \) is a path of length 2. In particular, if we let \( x = y \), then \( g(\Gamma(R)) \) is equal to or less than 3. \( \square \)

**Example 1** (See Example 1.5, p. 5 in [5]). Let

\[
R = \left\{ \begin{pmatrix} Z & \mathbb{Z}/2\mathbb{Z} \\ 0 & \mathbb{Z}/2\mathbb{Z} \end{pmatrix} \right\} \quad \text{and take } a = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \in R.
\]
Since the left annihilator of $a$ is equal to $\{0\}$ but the right annihilator of $a$ is not equal to $\{0\}$, $a$ is not a left zero-divisor, and so $a$ is an origin but $a$ is a right zero-divisor. Since there is no path from $a$ to $a^2$, $\Gamma(R)$ is not connected.

Let $S = \left\{ \left( \begin{array}{cc} Z & 0 \\ Z/2Z & Z/2Z \end{array} \right) \right\}$ and take $c = \left( \begin{array}{c} 2 \\ 0 \\ 0 \\ 1 \end{array} \right) \in S$.

Similarly, we note that $c$ is not a right zero-divisor, and so $c$ is a sink but $c$ is a left zero-divisor. Since there is also no path from $c^2$ to $c$, $\Gamma(S)$ is not connected.

**Remark 2.** In [3, Theorem 2.3], Anderson and Livingston have shown that for every commutative ring $R$, $\Gamma(R)$ is connected and $\text{diam}(\Gamma(R))$ is equal to or less than 3. But by Example 1 we can note that there is a noncommutative ring in which its zero-divisor graph is not connected and also note that the condition [there are no sources and no sinks in the zero-divisor graph of a noncommutative ring] is not superfluous to be connected.

**Theorem 2.3.** Let $R$ be a ring such that $X$ is a union of finite number of orbits under the left and right regular action on $X$ by $G$. Then $X = Z^*(R)$, and so $\Gamma_X(R)$ is connected and $\text{diam}(\Gamma_X(R))$ (resp. $g(\Gamma(R))$) is equal to or less than 3.

**Proof.** Since $X$ is a union of finite number of orbits under the left regular action on $X$ by $G$, then $Z^*_l(R) \subseteq Z^*_r(R) = X$ by [8, Theorem 2.2]. Similarly, we can show that if $X$ is a union of finite number of orbits under the right regular action on $X$ by $G$, then $Z^*_r(R) \subseteq Z^*_l(R) = X$. Thus $Z^*(R) = Z^*_l(R) = Z^*_r(R) = X$, which implies that $\Gamma(R)$ has no sources and no sinks, and so $\Gamma_X(R)$ is connected and $\text{diam}(\Gamma_X(R))$ (resp. $g(\Gamma(R))$) is equal to or less than 3 by Theorem 2.2. □

**Corollary 2.4.** Let $R$ be a ring such that $X \neq \emptyset$. If $X$ is finite, then $X = Z^*(R)$, and so $R$ is finite and $\left( |X| + 1 \right)^2 \geq |R|$.

**Proof.** Since $X \neq \emptyset$ and is finite, $X$ is a union of finite number of orbits under the left and right regular action on $X$ by $G$, and so we have $X = Z^*(R)$ by the argument given in the proof of Theorem 2.3. Hence $R$ is finite and then $\left( |X| + 1 \right)^2 \geq |R|$ by [11, Theorem I]. □

**Corollary 2.5.** Let $R$ be a finite ring. Then $\Gamma_X(R)$ is connected and $\text{diam}(\Gamma_X(R))$ (resp. $g(\Gamma(R))$) is equal to or less than 3.

**Proof.** Since $R$ is a finite ring, $X$ is a union of finite number of orbits under the left and right regular action on $X$ by $G$. Hence it follows from Theorem 2.3. □

**Proposition 2.6.** Let $n$ be any positive integer and $R$ be the matrix ring of all $n \times n$ matrices over a division ring $D$. Then $X = Z^*(R)$, and so $\Gamma_X(R)$ is connected and $\text{diam}(\Gamma_X(R))$ (resp. $g(\Gamma(R))$) is equal to or less than 3.
Proof. Let $x \in X$ be arbitrary. Then there exists $y \in X$ (resp. $z \in X$) such that $xy = 0$ (resp. $zx = 0$), which implies that $\text{ann}_r(x) \neq \emptyset$ (resp. $\text{ann}_l(x) \neq \emptyset$) for all $x \in X$, i.e., $X = Z^*(R)$. Hence $\Gamma_X(R)$ is connected and $\text{diam}(\Gamma(R))$ (resp. $y(\Gamma(R))$) is equal to or less than 3 by Theorem 2.2. □

Lemma 2.7. Let $R$ and $S$ be two rings. If $\Gamma(R)$ and $\Gamma(S)$ have no sources (resp. no sinks), then $\Gamma(R \times S)$ has no sources (resp. no sinks).

Proof. Let $(x_R, x_S) \in Z^*(R \times S)$ be arbitrary. Then $x_R \in Z^*(R)$ or $x_S \in Z^*(S)$. If $x_R \in Z^*(R)$, then there is $y_R \in X(R)$ such that $y_Rx_R = 0_R$ where $0_R$ is the additive identity of $R$ since $\Gamma(R)$ has no origins. Thus $(y_R, 0_S)(x_R, x_S) = (0_R, 0_S)$ where $0_S$ is the additive identity of $S$, and so $\Gamma(R \times S)$ has no sources. Similarly, if $x_S \in Z^*(S)$, then $\Gamma(R \times S)$ has no sources. By the similar argument, if $\Gamma(R)$ and $\Gamma(S)$ have no sinks, then $\Gamma(R \times S)$ has no sinks.

Corollary 2.8. Let $R_1, R_2, \ldots, R_n$ be rings for some positive integer $n$. If all $\Gamma(R_i)$ for $i = 1, 2, \ldots, n$ have no sources (resp. sinks), then $\Gamma(R_1 \times R_2 \times \cdots \times R_n)$ has no sources (resp. no sinks).

Proof. It follows from the Lemma 2.7 and the mathematical induction on $n$.

Proposition 2.9. Let $R$ be a ring with $X = o_r(x) \cup o_r(x^2) \cup \cdots \cup o_r(x^n)$ (resp. $X = o_l(x) \cup o_l(x^2) \cup \cdots \cup o_l(x^n)$) under the right (resp. left) regular action on $X$ by $G$ for some positive integer $n$. If $n = 1$ and $|X| \geq 3$, or $n = 2$ and $o_r(x^2) \neq \{x^2\}$, or $n = 3$ and $o_r(x^2) \neq \{x^2\}$ for some $i = 2$ or 3, or $n \geq 4$, then there exists a cycle of length 3 in $\Gamma(R)$.

Proof. Consider the right regular action of $G$ on $X$. If $n = 1$, right regular action is transitive, then $\Gamma(R)$ is complete by Proposition 2.1. Since $|X| \geq 3$, there exists a cycle of length 3 in $\Gamma(R)$. If $n = 2$ and $o_r(x^2) \neq \{x^2\}$, then there exists $g \in G$ such that $x^2g \neq x^2$. Since $X = o(x) \cup o(x^2)$ and $x^2g \in X$, $x^2g = hx$ or $hx^2$ for some $h \in G$. Thus $x^2 \longrightarrow x \longrightarrow x^2g \longrightarrow x^2$ is a cycle of length 3. If $n = 3$ and $o_r(x^2) \neq \{x^2\}$ for some $i = 2$ or 3, then there exists $g \in G$ such that $x^2g \neq x^2$. Since $X = o(x) \cup o(x^2) \cup o(x^3)$ and $x^2g \in X$, $x^2g = hx$ or $hx^2$ for some $h \in G$. Thus $x^2 \longrightarrow x^2 \longrightarrow x^2g \longrightarrow x^3$ is a cycle of length 3. Finally, if $n \geq 4$, then clearly $x^{n-2} \longrightarrow x^{n-1} \longrightarrow x^n \longrightarrow x^{n-2}$ is a cycle of length 3. Similarly, the result holds under the left regular action of $G$ on $X$.

Remark 3. Let $R$ be a ring. Then for each $x \in X$, $\text{ann}_l^r(x)$ (resp. $\text{ann}_r^l(x)$) is a union of orbits under the left (resp. right) regular action on $X$ by $G$. Indeed, let $y \in \text{ann}_l^r(x)$ be arbitrary. Then we have $o_r(y) \subseteq \text{ann}_l^r(x)$, and so $\bigcup_{y \in \text{ann}_l^r(x)} o_r(y) \subseteq \text{ann}_l^r(x)$. Clearly, $\text{ann}_l^r(x) \subseteq \bigcup_{y \in \text{ann}_l^r(x)} o_r(y)$. Hence $\text{ann}_l^r(x) = \bigcup_{y \in \text{ann}_l^r(x)} o_r(y)$, i.e., $\text{ann}_l^r(x)$ is a union of orbits under the left regular action on $X$ by $G$. By the similar argument, $\text{ann}_r^l(x)$ is a union of orbits under the right regular action on $X$ by $G$. 
Theorem 2.10. Let $R$ be a ring such that $X$ is a union of finite number of orbits under the left (resp. right) regular action on $X$ by $G$. If $R$ is a local ring, then there is a vertex of $\Gamma_X(R)$ which is adjacent to every other vertex in $\Gamma_X(R)$.

Proof. Let $X$ be a union of $n$ orbits under the left (resp. right) regular action on $X$ by $G$. Since $R$ is a local ring, by [8, Lemma 2.3] there exists $x \in X$ such that $x^n \neq 0 = x^{n+1}$ and $X = o_r(x) \cup o_l(x^2) \cup \cdots \cup o_l(x^n)$. Hence we have $\text{ann}_r(x^n) = X$, i.e., $a \rightarrow x^n$ for all $a \in X$, which means that $x^n$ is adjacent to every other vertex in $\Gamma_X(R)$. By the similar argument, we can show that if $X$ is a union of $n$ orbits under the right regular action on $X$ by $G$, then there exists $y \in X$ such that $y^n \neq 0 = y^{n+1}$ and $X = o_r(y) \cup o_r(y^2) \cup \cdots \cup o_r(y^n)$. Thus $\text{ann}_l(y^n) = X$, i.e., $y^n \rightarrow b$ for all $b \in X$, which means that $y^n$ is adjacent to every other vertex in $\Gamma_X(R)$.

Remark 4. We note that in the proof of Theorem 2.11 if $R$ is a local ring such that $X = o_r(x) \cup o_l(x^2) \cup \cdots \cup o_l(x^n)$ (resp. $X = o_r(x) \cup o_l(x^2) \cup \cdots \cup o_r(x^n)$) with $x^n \neq 0 = x^{n+1}$ under the left (resp. right) regular action on $X$ by $G$, then the subgraph $\Gamma_{o_r(x^n)}$ (resp. $\Gamma_{o_l(x^n)}$) of $\Gamma_X(R)$ is complete.

Corollary 2.11. If $R$ is a finite local ring, then there is a vertex of $\Gamma_X(R)$ which is adjacent to every other vertex in $\Gamma_X(R)$.

Proof. Since $R$ is a finite ring, $X$ is a union of finite number of orbits under the left and right regular action on $X$ by $G$. Hence it follows from Theorem 2.10.

Recall that a ring $R$ is called unit-regular if for every $x \in R$ there exists a unit $g \in R$ such that $xgx = x$. In [10], it has been shown that $R$ is a unit-regular ring if and only if for every orbit $o_r(x)$ ($x \in X$) under the left regular action on $X$ by $G$, there exists some idempotent $e \in X$ such that $o_r(x) = o_r(e)$. Similarly, we can show that $R$ is a unit-regular ring if and only if for every orbit $o_l(x)$ ($x \in X$) under the right regular action on $X$ by $G$, there exists some idempotent $e \in X$ such that $o_l(x) = o_l(e)$.

Proposition 2.12. Let $R$ be a unit-regular ring such that $X \neq \emptyset$. Then $\Gamma_X(R)$ is connected and $\text{diam}(\Gamma_X(R))$ (resp. $g(\Gamma(R))$) is equal to or less than 3.

Proof. Let $x \in X$ be arbitrary. Then there exists an idempotent $e_1 \in X$ such that $o_r(x) = o_r(e_1)$ under the left regular action on $X$ by $G$ by [10, Lemma 2.3]. By the similar argument, there exists an idempotent $e_2 \in X$ such that $o_l(x) = o_l(e_2)$ under the right regular action on $X$ by $G$. Hence there exists $g_1 \in G$ (resp. $g_2 \in G$) such that $x = g_1e_1$ (resp. $x = e_2g_2$). Since $x(1 - e_1) = g_1e_1(1 - e_1) = 0$ (resp. $(1 - e_2)x = (1 - e_2)e_2g_2 = 0$), $x$ is neither source nor sink. Thus $\Gamma_X(R)$ is connected and $\text{diam}(\Gamma_X(R))$ is equal to or less than 3 by Theorem 2.2. □
Proposition 2.13. Let $R$ be a unit-regular ring. Then $\Gamma_X(R)$ is complete if and only if the set of all idempotents in $R$ is orthogonal and the left regular action on $X$ by $G$ is trivial, i.e., $o_x(x) = \{x\}$ for all $x \in X$.

Proof. ($\Rightarrow$) Suppose that $\Gamma_X(R)$ is complete. Clearly, the set of all idempotents in $R$ is orthogonal. Assume that the left regular action of $G$ on $X$ is not trivial. Then there exists an idempotent $e \in X$ such that $o_x(e) \neq \{e\}$ by [10, Lemma 2.3] and so there exists $y \neq e \in o_x(e)$ such that $y = ge$ for some $g \in G$. Since $\Gamma_X(R)$ is complete and $y, e(y \neq e) \in X$, $0 = ye = (ge)e = ge = y$, a contradiction. Hence the left regular action on $X$ by $G$ is trivial.

($\Leftarrow$) It follows from [10, Lemma 2.3]. □

Corollary 2.14. Let $R$ be a unit-regular ring. Then $\Gamma_X(R)$ is complete if and only if the set of all idempotents in $R$ is orthogonal and the right regular action on $X$ by $G$ is trivial, i.e., $o_x(x) = \{x\}$ for all $x \in X$.

Proof. It follows from the similar argument given in the proof of Proposition 2.13. □

Lemma 2.15. Let $R$ be a ring. If under the left (resp. right) regular action on $X$ by $G$, $y \in o_x(x)$ (resp. $y \in o_r(x)$) for some $x \in X$, then $\text{ann}_r(x) = \text{ann}_r(y)$ (resp. $\text{ann}_l(x) = \text{ann}_l(y)$).

Proof. If $y \in o_x(x)$ (resp. $y \in o_r(x)$) for some $x \in X$, then there exists $g \in G$ (resp. $h \in G$) such that $y = gx$ (resp. $y = xh$). It is obvious to show that $\text{ann}_r(x) = \text{ann}_r(y)$ (resp. $\text{ann}_l(x) = \text{ann}_l(y)$). □

Corollary 2.16. Let $R$ be a unit-regular ring with $X \neq \emptyset$. Then for any $x \in X$ there exists an idempotent $e \in X$ such that $\text{ann}_r(x) = \text{ann}_r(e)$ (resp. $\text{ann}_l(x) = \text{ann}_l(e)$).

Proof. It follows from the Lemma 2.15 and [10, Lemma 2.3]. □

Proposition 2.17. Let $R$ be a unit-regular ring such that $X \neq \emptyset$ and $2 = 2 \cdot 1$ is a unit in $R$. Then there exists a cycle of length 4 in $\Gamma(R)$.

Proof. Let $e \in X$ be an idempotent. Since $2 = 2 \cdot 1 \in G$, $e \neq 1 - e, -e$. Thus $e \rightarrow 1 - e \rightarrow -e \rightarrow e - 1 \rightarrow e$ is a cycle of length 4 in $\Gamma(R)$. □

3. Automorphism of graph over $\text{Mat}_2(\mathbb{Z}_p)$

Recall that a graph automorphism $f$ of a graph $\Gamma(R)$ is a bijection $f : \Gamma(R) \rightarrow \Gamma_X(R)$ which preserves adjacency. Of course, the set $\text{Aut}(\Gamma(R))$ of all graph automorphisms of $\Gamma(R)$ forms a group under the usual composition of functions. In [3], Anderson and Livingston computed $\text{Aut}(\Gamma(\mathbb{Z}_n))$. In this section, we compute $\text{Aut}(\Gamma(\text{Mat}_2(\mathbb{Z}_p)))$ where $\text{Mat}_2(\mathbb{Z}_p)$ is the matrix ring of all $2 \times 2$ matrices over $\mathbb{Z}_p$ for any prime $p$. 
Lemma 3.1. Let $R$ be a ring and $f : \Gamma_X(R) \to \Gamma_X(R)$ be a graph automorphism of $\Gamma_X(R)$. Then for all $x \in X$, $f(\text{ann}_R(x)) = \text{ann}_R(f(x))$ (resp. $f(\text{ann}_R(x)) = \text{ann}_R(f(x))$).

Proof. Let $y \in f(\text{ann}_R(x))$ be arbitrary. Then $y = f(z)$ for some $z \in \text{ann}_R(x)$. Since $xy = 0$, $0 = f((xy)f(x)) = yf(x)$ and so $y \in \text{ann}_R(f(x))$. Hence $f(\text{ann}_R(x)) \subseteq \text{ann}_R(f(x))$. Let $z \in \text{ann}_R(f(x))$ be arbitrary. Then $zf(x) = 0$. Since $f$ is one to one, there exists $z_1 \in X$ such that $f(z_1) = z$. Then $0 = zf(x) = f(z_1)f(x) = f(z_1x)$, and so $z_1x = 0$. Since $z_1 \in \text{ann}_R(x)$ and $z = f(z_1) \in f(\text{ann}_R(x))$, $f(x) \subseteq f(\text{ann}_R(x))$. By the similar argument, we have $f(\text{ann}_R(x)) = \text{ann}_R(f(x))$.

In a ring $R$ with identity the left (resp. right) regular action of $G$ on $X$ is said to be half-transitive if $G$ is transitive on $X$ or if $o_\ell(x)$ (resp. $o_r(x)$) is a finite set with $|o_\ell(x)| > 1$ (resp. $|o_r(x)| > 1$) and $|o_\ell(x)| = |o_r(x)|$ (resp. $|o_r(x)| = |o_\ell(y)|$) for all $x$ and $y \in X$. In [9, Theorem 2.4 and Lemma 2.7], it was shown that if $R$ is a matrix ring of all $2 \times 2$ matrices over a finite field $F$, then $G$ is half-transitive on $X$ by the left (resp. right) regular action and $|o_\ell(x)| = |F|^2 - 1$ (resp. $|o_r(x)| = |F|^2 - 1$) for all $x \in X$.

Lemma 3.2. Let $p$ be a prime and $R = \text{Mat}_2(\mathbb{Z}_p)$. Then for any $x \in X$, $\text{ann}_R(x) = o_\ell(y)$ (resp. $\text{ann}_R(x) = o_r(z)$) for some $y \in X$ (resp. $z \in X$).

Proof. By [9, Lemma 2.7], we have $|o_\ell(x)| = p^2 - 1$ (resp. $|o_r(x)| = p^2 - 1$) for all $x \in X$. Since $\text{ann}_R(x)$ (resp. $\text{ann}_R(x)$) is a union of a finite number of orbits under the left (resp. right) regular action of $G$ on $X$ by Remark 3 and since the left (resp. right) regular action of $G$ on $X$ is half-transitive by [9, Theorem 2.4], $|o_r(y)|$ (resp. $|o_\ell(z)|$) for all $y \in \text{ann}_R(x)$ (resp. all $z \in \text{ann}_R(x)$) is a divisor of $|\text{ann}_R(x)|$ (resp. $|\text{ann}_R(x)|$) and then $|\text{ann}_R(x)| = p^2 - 1$ or $p^3 - 1$ (resp. $|\text{ann}_R(x)| = p^2 - 1$ or $p^3 - 1$) since $|\text{ann}_R(x)| = p^2$ or $p^3$ (resp. $|\text{ann}_R(x)| = p^2$ or $p^3$) and so $|\text{ann}_R(x)| = p^2 - 1$ (resp. $|\text{ann}_R(x)| = p^2 - 1$). Hence we have the result.

Lemma 3.3. Let $p$ be a prime and $R = \text{Mat}_2(\mathbb{Z}_p)$. Then the number of orbits under the left (resp. right) regular action on $X$ by $G$ is $p^2 + 1$.

Proof. Let $\mu$ be the number of orbits under the left (resp. right) regular action on $X$ by $G$. Note that $|G| = (p^2 - 1)(p^2 - p)$. Thus $|X| = |R| - |G| - 1 = p^2 - (p^2 - 1)(p^2 - p) - 1 = (p + 1)(p^2 - 1)$. Since the cardinality of any orbit under the left (resp. right) regular action on $X$ by $G$ is $p^2 - 1$ by [9, Lemma 2.7], $\mu = |X|/(p^2 - 1) = p + 1$.

Lemma 3.4. Let $p$ be a prime, $R = \text{Mat}_2(\mathbb{Z}_p)$ and let $N(p)$ be the set of nonzero nilpotents in $R$. Then $|N(p)| = p^2 - 1$.

Proof. Let

$$N_1(p) = \left\{ \begin{pmatrix} a & b \\ \alpha a & \alpha b \end{pmatrix} \in N(p) \mid a, b, \alpha \neq 0 \right\}$$
and
\[ N_2(p) = \left\{ \begin{pmatrix} a & -\alpha a \\ b & -\alpha b \end{pmatrix} \in N(p) \mid a, b, \alpha \neq 0 \right\}. \]
We will show that \( N_1(p) = N_2(p) \). Let
\[ \begin{pmatrix} a & -\alpha a \\ b & -\alpha b \end{pmatrix} \in N_2(p) \]
be arbitrary. Since \( A^2 = 0 \) and \( a, b \neq 0 \), we have
\[ A = \begin{pmatrix} \alpha b & -\alpha^2 b \\ b & -\alpha b \end{pmatrix} \in N_2(p), \]
and also \( (1/\alpha^2) \begin{pmatrix} \alpha b & -\alpha^2 b \\ b & -\alpha b \end{pmatrix} \in N_2(p) \).
Since
\[ (1/\alpha^2) \begin{pmatrix} \alpha b & -\alpha^2 b \\ b & -\alpha b \end{pmatrix} = \begin{pmatrix} (-1/\alpha)(-b) & -b \\ (-1/\alpha^2)(-b) & (-1/\alpha)(-b) \end{pmatrix} \in N_1(p), \]
we have \( N_2(p) \subseteq N_1(p) \). By the similar argument, we can have \( N_1(p) \subseteq N_2(p) \).

Let \( A \) be any nonzero nilpotent in \( R \). Then
\[ A = \begin{pmatrix} a & b \\ \alpha a & \alpha b \end{pmatrix} \text{ or } \begin{pmatrix} a & \alpha a \\ b & \alpha b \end{pmatrix} \]
for some \( \alpha \in \mathbb{Z}_p \).

Note that since \( A \) is a nonzero nilpotent in \( R \), \( b \neq 0 \). Consider the following cases:

**Case 1.** \( \alpha = 0 \);
Since
\[ A^2 = 0, \quad A = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \]
for all nonzero \( b \in \mathbb{Z}_p \).

**Case 2.** \( \alpha \neq 0 \);
In this case, \( a \neq 0 \). Hence we have \( N_1(p) = N_2(p) \) by the above argument.
Since \( A^2 = 0 \), we have \( A = \begin{pmatrix} -\alpha b & b \\ -\alpha^2 b & \alpha b \end{pmatrix} \).

Consequently, we have
\[
|N(p)| = |N_1(p)| + \left| \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in N(p) : b(\neq 0) \right\} \right| \\
+ \left| \left\{ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \in N(p) : b(\neq 0) \right\} \right| \\
= (p-1)(p-1) + 2(p-1) = p^2 - 1.
\]
\[ \square \]
Example 2. Let \( R = \text{Mat}_2(\mathbb{Z}_2) \). Then \( X = \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \} \), where
\[
\begin{align*}
x_1 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\
x_2 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\
x_3 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
x_4 &= \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \\
x_5 &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \\
x_6 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
x_7 &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \\
x_8 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\
x_9 &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\end{align*}
\]

Note that \( \{ x_2, x_4, x_9 \} \) is the set of nonzero nilpotents in \( R \). Under the left (resp. right) regular action on \( X \) by \( G \), there are three orbits \( o_1(x_2) = \{ x_2, x_6, x_7 \} \), \( o_2(x_4) = \{ x_1, x_4, x_5 \} \), \( o_3(x_9) = \{ x_3, x_8, x_9 \} \) (resp. \( o_1(x_2) = \{ x_1, x_2, x_3 \} \), \( o_2(x_4) = \{ x_4, x_5, x_8 \} \), \( o_3(x_9) = \{ x_5, x_7, x_9 \} \)).

We can compute \( \text{Aut}(\Gamma(R)) = \{1, f, g \circ f, f \circ g, g \circ f \circ g \} \), where
\[
\begin{align*}
f &= \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ x_3 & x_2 & x_1 & x_9 & x_7 & x_5 & x_8 & x_6 & x_4 \end{pmatrix}, \\
g &= \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ x_3 & x_4 & x_8 & x_2 & x_1 & x_3 & x_7 & x_5 & x_9 \end{pmatrix},
\end{align*}
\]
are permutations.

Observe that \( \text{Aut}(\Gamma(R)) \) is isomorphic to \( S_3 \), the symmetric group of degree 3.

Theorem 3.5. Let \( p \) be a prime and \( R = \text{Mat}_2(\mathbb{Z}_p) \). Then \( \text{Aut}(\Gamma(R)) \neq \{1\} \).

Proof. If \( p = 2 \), then \( \text{Aut}(\Gamma(R)) \neq \{1\} \) by Example 2. Suppose that \( p \geq 3 \). Let \( N(p) \) be the set of nonzero nilpotents in \( R \). Since the number orbits is \( p + 1 \) by Lemma 3.3 under the left (resp. right) regular action on \( X \) by \( G \) and \( |N(p)| = p^2 - 1 \) by Lemma 3.4, there exists \( x \in X \) such that \( |o(x) \cap N(p)| \geq 2 \). Let \( x_1, x_2 \in o_1(x) \cap N(p) \) (\( x_1 \neq x_2 \)). Since \( x_1 \) and \( x_2 \) are nilpotents, we have \( \text{ann}_{\ell}(x_1) = o_1(x_1) = \text{ann}_{\ell}(x_2) \) by Lemma 3.2. We have also \( \text{ann}_{r}(x_1) = \text{ann}_{r}(x_2) \). Indeed, if \( a \in \text{ann}_{\ell}(x_1) \), then \( 0 = x_1a = gx_2a = 0 \) for some \( g \in G \) since \( x_2 \in o_1(x_1) \), which implies that \( a \in \text{ann}_{r}(x_2) \), and so \( \text{ann}_{\ell}(x_1) \subseteq \text{ann}_{r}(x_2) \). By the similar argument, we have \( \text{ann}_{r}(x_2) \subseteq \text{ann}_{\ell}(x_1) \). Also we have \( \text{ann}_{\ell}(x_1) = o_1(x_1) = o_2(x_2) = \text{ann}_{r}(x_2) \) by Lemma 3.2. Let \( f = (x_1, x_2) \) be a transposition in \( S_{|X|} \), the symmetric group of degree \( |X| \). Since \( x_1 \neq x_2, f \neq 1 \). We will show that \( f \in \text{Aut}(\Gamma(R)) \). Consider \( x_1y = z_1 \) for some \( y \in X \). If \( y = x_1 \), then \( f(x_1)f(y) = x_2x_2 = 0 \). If \( y = x_2 \), then \( f(x_1)f(y) = x_2x_1 = g_1x_1x_1 = 0 \) for some \( g_1 \in G \) since \( x_2 \in o_1(x_1) \). If \( y \neq x_1, x_2 \), then \( f(x_1)f(y) = x_2y = g_1y = 0 \) for some \( g_1 \in G \) since \( x_2 \in o_1(x_1) \). Also consider \( xz_2 = 0 \) for some \( z \in X \). If \( z = x_1 \), then \( f(z)f(x_1) = x_2x_2 = 0 \). If \( z = x_2 \), then \( f(z)f(x_1) = x_1x_2 = h_1x_2x_2 = 0 \) for some \( h_1 \in G \) since \( x_1 \in o(x_2) \). If \( z \neq x_1, x_2 \), then \( f(z)f(x_1) = xz_2 = xz_1h_2 = 0 \) for some \( h_2 \in G \) since \( x_2 \in o_1(x_1) \). Consequently, \( f \in \text{Aut}(\Gamma(R)) \), and so \( \text{Aut}(\Gamma(R)) \neq \{1\} \). □

Remark 5. Let \( p \) be a prime, \( R = \text{Mat}_2(\mathbb{Z}_p) \) and \( N(p) \) be the set of nonzero nilpotents in \( R \). We can choose that \( f(\neq 1) \in \text{Aut}(\Gamma(R)) \) by Theorem 3.5. Then we note that \( (1) \ f(a) \in N(p) \) for all \( a \in N(p) \); \( (2) \) since \( f \) is bijective
Let as given in the proof of Lemma 3.6, Lemma 3.8.

\[ a \cup \cdots \cup b \]

be a prime, \( R = \text{Mat}_2(\mathbb{Z}_p) \) and \( N(p) \) be the set of all nonzero nilpotents in \( R \). Then under the left (resp. right) regular action on \( X \) by \( G \), \( o_\ell(a) \cap N(p) = o_\ell(a) \cap N(p) = o_\ell(a) \cap o_\ell(a) \) for all \( a \in N(p) \).

**Proof.** Let \( b \in o_\ell(a) \cap N(p) \) be arbitrary. Since \( o_\ell(a) = o_\ell(b) \), \( ba = ab = 0 \), and thus \( b \in \text{ann}^*_a(a) = o_\ell(a) \). Hence \( o_\ell(a) \cap N(p) \subseteq o_\ell(a) \cap N(p) \) and \( o_\ell(a) \cap N(p) \subseteq o_\ell(a) \cap o_\ell(a) \). By the similar argument, we have \( o_\ell(a) \cap N(p) \subseteq o_\ell(a) \cap N(p) \) and \( o_\ell(a) \cap N(p) \subseteq o_\ell(a) \cap o_\ell(a) \).

Therefore, \( o_\ell(a) \cap N(p) = o_\ell(a) \cap N(p) \subseteq o_\ell(a) \cap o_\ell(a) \). By Remark 4, we already knew that \( |o_\ell(a) \cap N(p)| = |o_\ell(a) \cap N(p)| = p - 1 \). Next, we will show that \( o_\ell(a) \cap N(p) = o_\ell(a) \cap o_\ell(a) \). Let \( S = \text{ann}_a(a) \cap \text{ann}_p(a) \). Then \( S = (o_\ell(a) \cap o_\ell(a)) \cup \{0\} \).

Since \( S \) is an additive subgroup of \( \text{ann}_a(a) \) and \( |\text{ann}_p(a)| = p^2 \), \( |S| = 1 \) or \( p \). Since \( |o_\ell(a) \cap o_\ell(a)| \geq |o_\ell(a) \cap N(p)| = p - 1 \geq 2 \), \( |S| = |o_\ell(a) \cap o_\ell(a)| + 1 \geq 2 \), and thus \( |S| = p \).

Since \( |o_\ell(a) \cap o_\ell(a)| = |S| - 1 = p - 1 = |o_\ell(a) \cap N(p)| = |o_\ell(a) \cap N(p)| \) and \( o_\ell(a) \cap N(p) \subseteq o_\ell(a) \cap o_\ell(a) \), we have \( |o_\ell(a) \cap N(p)| = |o_\ell(a) \cap o_\ell(a)| = |o_\ell(a) \cap o_\ell(a)| \).

**Remark 6.** Let \( p \) be a prime, \( R = \text{Mat}_2(\mathbb{Z}_p) \) and \( N(p) \) be the set of nonzero nilpotents in \( R \). We can choose \( a_1, \ldots, a_{p+1} \in N(p) \) such that \( X = o_\ell(a_1) \cup \cdots \cup o_\ell(a_{p+1}) \) (resp. \( X = \text{ann}_a(a_1) \cap \cdots \cap o_\ell(a_{p+1}) \)). Note that for each \( i = 1, \ldots, p+1 \), \( o_\ell(a_i) = o_\ell(a_i) \cap X = o_\ell(a_i) \cap \{o_\ell(a_1) \cap \cdots \cap o_\ell(a_{p+1})\} = [o_\ell(a_i) \cap o_\ell(a_1)] \cup \cdots \cup [o_\ell(a_i) \cap o_\ell(a_{p+1})] \).

**Lemma 3.7.** Let \( p \) be a prime, \( R = \text{Mat}_2(\mathbb{Z}_p) \) and \( N(p) \) be the set of nonzero nilpotents in \( R \). Consider \( X = o_\ell(a_1) \cup \cdots \cup o_\ell(a_{p+1}) \) (resp. \( X = \text{ann}_a(a_1) \cup \cdots \cup o_\ell(a_{p+1}) \)) for some \( a_1, \ldots, a_{p+1} \in N(p) \) as mentioned in Remark 6. Then under the left (resp. right) regular action on \( X \) by \( G \), \( |o_\ell(a_i) \cap o_\ell(a_j)| = p - 1 \) for all \( a_i, a_j \in N(p) \) (i, j = 1, \ldots, p+1).

**Proof.** Let \( A_{ij} = \text{ann}_a(a_i) \cap \text{ann}_p(a_j) \) for all \( i, j = 1, \ldots, p+1 \). Note that \( A_{ij} = |\text{ann}_p(a_i) \cap o_\ell(a_j)| \cup \{0\} \). If \( i = j \), then \( |\text{ann}_p(a_i) \cap o_\ell(a_j)| = p - 1 \) as given in the proof of Lemma 3.6. Suppose that \( i \neq j \). Since \( A_{ij} \) is an additive subgroup of \( \text{ann}_p(a_i) \) with \( |\text{ann}_p(a_i)| = p^2 \), \( |A_{ij}| = 1 \) or \( p \). Hence \( |o_\ell(a_i) \cap o_\ell(a_j)| = 0 \) or \( p - 1 \).

Assume that \( |A_{ij}| = 1 \) (equivalently, \( |o_\ell(a_i) \cap o_\ell(a_j)| = 0 \)) for some \( i, j \). Then \( |A_{ik}| > |A_{ij}| \) for some \( k \). Since \( |A_{ij}| = p \) (equivalently, \( |o_\ell(a_i) \cap o_\ell(a_j)| = p - 1 \)) as given in the proof of Lemma 3.6, \( |A_{ik}| > p \), a contradiction. Therefore, \( |A_{ij}| = p \), and so \( |o_\ell(a_i) \cap o_\ell(a_j)| = p - 1 \) for all \( i, j = 1, \ldots, p+1 \). □

**Lemma 3.8.** Let \( p \) be a prime, \( R = \text{Mat}_2(\mathbb{Z}_p) \) and \( N(p) \) be the set of nonzero nilpotents in \( R \). Consider \( X = o_\ell(a_1) \cup \cdots \cup o_\ell(a_{p+1}) \) (resp. \( X = \text{ann}_a(a_1) \cup \cdots \cup o_\ell(a_{p+1}) \)) for some \( a_1, \ldots, a_{p+1} \in N(p) \) as mentioned in Remark 5. If
\[ s_j = (1, j) \text{ is a transposition in } S_{p+1}, \text{ the symmetric group of degree } p + 1, \text{ and} \]
\[ f_{s_j} : \Gamma(R) \rightarrow \Gamma(R) \text{ is a bijective map such that } f_{s_j}(o_{t(a_i)}) = o_{t(s_j(a_i))}, \text{ then} \]
\[ f_{s_j} \text{ is a graph automorphism in } \Gamma(R). \]

**Proof.** Note that since \( f_{s_j} : \Gamma(R) \rightarrow \Gamma(R) \) is a bijective map such that \( f_{s_j}(o_{t(a_i)}) = o_{t(s_j(a_i))}, f_{s_j}(o_{t(a_i) \cap o_{t(a_k)})} = o_{t(s_j(a_i) \cap a_{s_j(a_k)})} \) for all \( i, k = 1, \ldots, p + 1. \)

Let \( x, y \in X \) be arbitrary. Consider the following cases.

**Case 1.** \( x, y \in o_t(a_1) \cap o_t(a_1). \)

Since \( a_i^2 = 0, xy = yx = 0. \) Note that \( f_{s_j}(x), f_{s_j}(y) \in o_t(a_j) = o_t(a_j), \) and so \( f_{s_j}(x)f_{s_j}(y) = f_{s_j}(xy) = f_{s_j}(0) = 0 \) and also \( f_{s_j}(y)f_{s_j}(x) = 0. \)

**Case 2.** \( x, y \in o_t(a_1) \cap o_t(a_j). \)

By the similar argument given to the case 1, \( xy = yx = 0 \) and also \( f_{s_j}(x)f_{s_j}(y) = f_{s_j}(y)f_{s_j}(x) = 0. \)

**Case 3.** \( x \in o_t(a_1) \cap o_t(a_1), y \in o_t(a_1) \cap o_t(a_j) (j \neq 1). \)

Then \( xy = 0. \) Note that \( f_{s_j}(x) \in o_t(a_j) \cap o_t(a_j), f_{s_j}(y) \in o_t(a_1) \cap o_t(a_1), \) and so \( f_{s_j}(y)f_{s_j}(x) = 0. \) Assume that \( xy = 0. \) Then \( a_1a_j = 0, \) which implies that \( o_t(a_1) = o_t(a_j), \) a contradiction. Hence \( xy \neq 0. \) Assume that \( f_{s_j}(x)f_{s_j}(y) = 0. \)

Since \( f_{s_j}(x) \in o_t(a_j) \cap o_t(a_1), f_{s_j}(y) \in o_t(a_1) \cap o_t(a_1), a_ja_1 = 0, \) which implies that \( o_t(a_1) = o_t(a_j), \) also a contradiction. Hence we have \( f_{s_j}(x)f_{s_j}(y) \neq 0. \)

**Case 4.** \( x \in o_t(a_1) \cap o_t(a_j), y \in o_t(a_1) \cap o_t(a_1). \)

By the similar argument given to the case 3, \( xy = 0 \) and also \( f_{s_j}(x)f_{s_j}(y) = 0; \)
\[ yx \neq 0 \text{ and } f_{s_j}(y)f_{s_j}(x) \neq 0. \]

**Case 5.** \( x \in o_t(a_1) \cap o_t(a_1), y \in o_t(a_1) \cap o_t(a_k) (i, k \neq 1, j). \)

Then \( x = g_1a_1 = a_1h_1, y = g_2a_1 = a_kh_2 \text{ for some } g_1, g_2, h_1, h_2 \in G. \) If \( xy = 0, \) then \( a_1a_k = 0, \) which implies that \( o_t(a_1) = o_t(a_k), \) a contradiction. Hence we have \( xy \neq 0. \) Since \( f(x) \in o_t(a_j) \cap o_t(a_1), f(y) \in o_t(a_1) \cap o_t(a_k), \) we also have \( f(x)f(y) \neq 0. \) Similarly, we have \( yx \neq 0 \) and \( f(y)f(x) \neq 0. \)

**Case 6.** \( x \in o_t(a_i) \cap o_t(a_r), y \in o_t(a_k) \cap o_t(a_s) (i, k, r, s \neq 1, j). \)

If \( xy = 0, \) then \( a_ia_k = 0. \) Since \( f(x) \in o_t(a_i) \cap o_t(a_r), f(y) \in o_t(a_k) \cap o_t(a_s), \)
\[ f(x)f(y) = 0. \] Similarly we have that if \( yx = 0, f(y)f(x) = 0. \)

Consequently, \( f_{s_j} \) is a graph automorphism in \( \Gamma(R). \)

**Theorem 3.9.** Let \( p \) be a prime and let \( R = \text{Mat}_2(\mathbb{Z}_p) \). Then \( \text{Aut}(\Gamma(R)) \simeq S_{p+1} \) where \( S_{p+1} \) is the symmetric group of degree \( p + 1. \)

**Proof.** Let \( N(p) \) be the set of nonzero nilpotents in \( R. \) We can choose \( a_1, \ldots, a_{p+1} \in N(p) \) such that \( X = o_t(a_1) \cup \cdots \cup o_t(a_{p+1}). \) Define \( \sigma : S_{p+1} \rightarrow \text{Aut}(\Gamma(R)) \) by \( \sigma(s) = f_s \) for all \( s \in S_{p+1} \) where \( f_s(o_t(a_i)) = o_t(s(a_i)) \) for all \( i = 1, \ldots, p + 1. \) Then \( \sigma \) is well-defined and onto. Indeed, by Lemma 3.1 and Lemma 3.2, we have that if \( f \in \text{Aut}(\Gamma(R)) \) is arbitrary, then for all \( i = 1, \ldots, p + 1, f(o_t(a_i)) = o_t(s(a_i)) \) for some \( s \in S_{p+1}. \) Since \( S_{p+1} \) is generated by the \( p \) transpositions \( s_1 = (1, 2), \ldots, s_p = (1, p + 1), \) and \( f_{s_1}, \ldots, f_{s_p} \in \text{Aut}(\Gamma(R)) \) are graph automorphisms.
Aut(\Gamma(R)) by Lemma 3.8, Aut(\Gamma(R)) is generated by the \( p \) graph automorphisms \( f_{s_1}, \ldots, f_{s_p} \in \text{Aut}(\Gamma(R)) \) where \( f_{s_j}(\alpha_i)=\alpha_{s_j(i)} \) for all \( i=1, \ldots, p+1 \) and \( j=1, \ldots, p \). Thus \( \lvert S_{p+1} \rvert = \lvert \text{Aut}(\Gamma(R)) \rvert \), which implies that \( \sigma \) is a bijective map. Also \( \sigma \) is a group homomorphism by observing that for all \( s_i, s_j \in S_{p+1} \) \((i, j=1, \ldots, p)\) and all \( \alpha_i(\alpha_k)(k=1, \ldots, p+1)\), \((f_{s_i} \circ f_{s_j})(\alpha_i(\alpha_k)) = f_{s_i s_j}(\alpha_i(\alpha_k))\). Therefore, \( \text{Aut}(\Gamma(R)) \simeq S_{p+1} \). □

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