EVALUATING SOME DETERMINANTS OF MATRICES WITH RECURSIVE ENTRIES

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Abstract. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $\beta = (\beta_1, \beta_2, \ldots)$ be two sequences with $\alpha_1 = \beta_1$ and $k$ and $n$ be natural numbers. We denote by $A^{(k, \pm)}_{\alpha, \beta}(n)$ the matrix of order $n$ with coefficients $a_{i,j}$ by setting $a_{1,i} = \alpha_i$, $a_{i,1} = \beta_i$ for $1 \leq i \leq n$ and $a_{i,j} = \begin{cases} a_{i-1,j-1} + a_{i-1,j} & \text{if } j \equiv 2, 3, 4, \ldots, k+1 \pmod{2k} \\ a_{i-1,j-1} - a_{i-1,j} & \text{if } j \equiv k+2, \ldots, 2k+1 \pmod{2k} \end{cases}$ for $2 \leq i, j \leq n$. The aim of this paper is to study the determinants of such matrices related to certain sequences $\alpha$ and $\beta$, and some natural numbers $k$.

1. Introduction

In [1], R. Bacher considered the determinants of matrices associated to the Pascal triangle. Furthermore, he introduced the generalized Pascal triangles as follows. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $\beta = (\beta_1, \beta_2, \ldots)$ be two sequences starting with a common first term $\alpha_1 = \beta_1$. Define a matrix $P_{\alpha, \beta}(n)$ of order $n$ with coefficients $p_{i,j}$ by setting $p_{1,i} = \beta_i$, $p_{i,1} = \alpha_i$ for $1 \leq i \leq n$ and $p_{i,j} = p_{i-1,j} + p_{i,j-1}$ for $2 \leq i, j \leq n$. The infinite matrix $P_{\alpha, \beta}(\infty)$ is called the generalized Pascal triangle associated to the sequences $\alpha$ and $\beta$. In addition he investigated some other similar constructions and made many interesting observations and posed some conjectures. Some of his conjectures were thoroughly investigated in [3] with positive answers.

In constructing the generalized Pascal triangles or the other similar constructions in which the coefficients, except for the first row and column, are determined by a recursive relation, only one recursive relation is used. Here we are willing to construct some similar arrangements associated to two arbitrary sequences $\alpha$ and $\beta$ being in the first row and column, respectively, and the remaining coefficients are determined by two different recursive relations. Let us define this more precisely as follows.

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Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $\beta = (\beta_1, \beta_2, \ldots)$ be two sequences starting with a common first term $\alpha_1 = \beta_1$ and $k$ be a natural number. Define a matrix $A_{\alpha, \beta}^{(k, \pm)}(n)$ of order $n$ with coefficients $a_{i,j}$ by setting $a_{i,1} = \beta_i$, $a_{1,i} = \alpha_i$ for $1 \leq i \leq n$ and

$$a_{i,j} = \begin{cases} a_{i-1,j-1} + a_{i-1,j} & \text{if } j \equiv 2, 3, 4, \ldots, k + 1 \pmod{2k} \\ a_{i-1,j-1} - a_{i-1,j} & \text{if } j \equiv k + 2, \ldots, 2k + 1 \pmod{2k} \end{cases}$$

for $2 \leq i, j \leq n$. When $k = 1$, we put $A_{\alpha, \beta}^{(1, \pm)}(n) = A_{\alpha, \beta}^{(1, 1)}(n)$.

In general, we are interested in the sequence of the determinants

$$(\det A_{\alpha, \beta}^{(k, \pm)}(1), \det A_{\alpha, \beta}^{(k, \pm)}(2), \ldots, \det A_{\alpha, \beta}^{(k, \pm)}(n), \ldots),$$

where $\alpha$ and $\beta$ are certain sequences having a common first entry.

On the other hand, when we consider the constant sequence $\alpha = (1, 1, 1, \ldots)$, we notice that the generalized Pascal triangle $P_{\alpha, \alpha}(\infty)$ is, in fact, the classical Pascal triangle. Hence, in the early studies, we restrict our investigation to this sequence $\alpha = (1, 1, 1, \ldots)$ only, and we consider the principal minors of infinite matrices $A_{\alpha, \alpha}^{(k, \pm)}(\infty)$.

In this research, it has been tried to prove three theorems,

**Theorem 1.1.** The matrices $A_{\alpha, \alpha}^{(\pm)}(n)$ associated to the sequence $\alpha = (1, 1, \ldots)$ have the determinant $3^{l-2}l!$ for every natural number $n$. In other words, we have

$$\det A_{\alpha, \alpha}^{(\pm)}(n) = \begin{cases} 3^{l-1} & \text{if } n = 2l, \quad (l = 1, 2, \ldots) \\ 3^l & \text{if } n = 2l + 1, \quad (l = 0, 1, 2, \ldots) \end{cases}$$

**Theorem 1.2.** The sequence $(\det A_{\alpha, \alpha}^{(2, \pm)}(n))$ of determinants associated to the sequence $\alpha = (1, 1, 1, \ldots)$ satisfies the following

$$\det A_{\alpha, \alpha}^{(2, \pm)}(n) = \begin{cases} (-5)^{2l-1} & \text{if } n = 4l, \quad (l = 1, 2, \ldots) \\ (-5)^{2l} & \text{if } n = 4l + r, \quad (r = 1, 2, 3; \quad l = 0, 1, 2, \ldots) \end{cases}$$

**Theorem 1.3.** The sequence $(\det A_{\alpha, \alpha}^{(3, \pm)}(n))$ of determinants associated to the sequence $\alpha = (1, 1, 1, \ldots)$ satisfies the following

$$\det A_{\alpha, \alpha}^{(3, \pm)}(n) = \begin{cases} 11^{3l-1} & \text{if } n = 6l, \quad (l = 1, 2, \ldots) \\ 11^3 & \text{if } n = 6l + r, \quad (r = 1, 2, 3, 4; \quad l = 0, 1, 2, \ldots) \\ 11^{3l+1} & \text{if } n = 6l + 5, \quad (l = 0, 1, 2, \ldots) \end{cases}$$

Here, we have the following conjecture:

**Conjecture.** Let $k$ and $n$ be natural numbers and $n - 1 = rk + s$ for some $r, s$ with $r \geq 0$ and $0 \leq s < k$. Let $\alpha = (1, 1, 1, \ldots)$. Then we have

$$\det A_{\alpha, \alpha}^{(k, \pm)}(n) = \begin{cases} \omega^{rk/2} & \text{if } r \text{ is even,} \\ \omega^{k(r-1)/2+s} & \text{if } r \text{ is odd,} \end{cases}$$

where $\omega = [1 - (-2)^{k+2}]/3$. 
2. Main results

As we mentioned before, we should concentrate on the sequence of determinants

$$(\det A_{\alpha,\alpha}^{(k, \pm)}(1), \det A_{\alpha,\alpha}^{(k, \pm)}(2), \ldots, \det A_{\alpha,\alpha}^{(k, \pm)}(n), \ldots)$$

for certain $k$. Therefore, in order to start, we consider the case $k = 1$, and prove the following theorem.

**Theorem 1.** The matrices $A_{\alpha,\alpha}^{\pm}(n)$ associated to the sequence $\alpha = (1, 1, 1, \ldots)$ have the determinant $3^{\frac{n-1}{2}}$ for every natural number $n$.

**Proof.** We apply LU-factorization method (see [2]). We claim that

$$A_{\alpha,\alpha}^{\pm}(n) = L \cdot U,$$

where $L = A_{\beta,\alpha}^{\pm}(n)$ with $\beta = (1, 0, 0, 0, \ldots)$, and where

$$U = \begin{bmatrix}
U_1 \\
U_2 \\
\vdots \\
U_n
\end{bmatrix},$$

with

$$U_i = \begin{cases}
(1, 1, 1, \ldots, 1) & \text{if } i = 1, \\
(0, 0, \ldots, 0, 1, -1, 1, -1, \ldots, u_{i,n-1}, u_{i,n}) & \text{if } i \equiv 0, \\
(0, 0, \ldots, 0, 3, -1, 3, -1, \ldots, u_{i,n-1}, u_{i,n}) & \text{if } i > 1 \text{ and } i \equiv 1,
\end{cases}$$

and $(u_{i,n-1}, u_{i,n})$ is satisfied in Table 1.

<table>
<thead>
<tr>
<th>$i \backslash n$</th>
<th>$n \equiv 0$</th>
<th>$n \equiv 1$</th>
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<tbody>
<tr>
<td>$i \equiv 0$</td>
<td>$(-1, 1)$</td>
<td>$(1, -1)$</td>
</tr>
<tr>
<td>$i \equiv 1$</td>
<td>$(3, -1)$</td>
<td>$(-1, 3)$</td>
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The matrix $L$ is a lower triangular matrix with 1’s on the diagonal, whereas $U$ is an upper triangular matrix with diagonal entries.
if \( n = 1 \),
1, 1
if \( n = 2 \),
1, 1, 3, 1, 3, 1, 3, \ldots , 1, 3
n-1 times (2-periodic)
1, 1, 3, 1, 3, 1, 3, \ldots , 3, 1
n-1 times (2-periodic)

Since \( \det L = 1 \) and \( \det U = 3^{\lfloor \frac{n+1}{2} \rfloor} \), it is obvious that the claimed factorization of \( A_{\alpha,\alpha}^\pm(n) \) immediately implies the validity of the theorem.

Suppose that
\[
L = (l_{i,j})_{1 \leq i,j \leq n} \quad \text{and} \quad U = (u_{i,j})_{1 \leq i,j \leq n}.
\]
Then by definition, we have
\[
l_{1,1} = 1, \quad l_{1,j} = 0, \quad l_{i,1} = 1 \quad \text{for} \quad 2 \leq i,j \leq n
\]
and
\[
(1) \quad l_{i,j} = \begin{cases} 
  l_{i-1,j-1} + l_{i-1,j} & \text{if} \quad j \equiv 0 \\
  l_{i-1,j-1} - l_{i-1,j} & \text{if} \quad j \equiv 1
\end{cases}
\]
for \( 2 \leq i,j \leq n \). Also we have
\[
(2) \quad (u_{1,j}, u_{2,j}, \ldots , u_{n,j})^T = \begin{cases} 
  (1, 0, 0, \ldots , 0)^T & j = 1, \\
  (1, -1, 1, -1, \ldots , -1, 1, 0, \ldots , 0)^T & j \equiv 0, \\
  (1, -1, 3, -1, 3, \ldots , -1, 3, 0, \ldots , 0)^T & j \equiv 0.
\end{cases}
\]

For the proof of the claimed factorization we compute the \((i,j)\)-entry of \( L \cdot U \), that is
\[
(L \cdot U)_{i,j} = \sum_{k=1}^{n} l_{i,k} u_{k,j}.
\]
It is easy to see that it is enough to show that \((L \cdot U)_{1,j} = 1, (L \cdot U)_{i,1} = 1\) for \( 1 \leq i,j \leq n \) and
\[
(3) \quad (L \cdot U)_{i,j} = \begin{cases} 
  (L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j} & j \equiv 0 \\
  (L \cdot U)_{i-1,j-1} - (L \cdot U)_{i-1,j} & j \equiv 1
\end{cases}
\]
for \( 2 \leq i,j \leq n \), in order to prove the theorem.

Let us do the required calculations. First, suppose that \( i = 1 \). Then
\[
(3) \quad (L \cdot U)_{1,j} = \sum_{k=1}^{n} l_{1,k} u_{k,j} = l_{1,1} u_{1,j} = 1.
\]
Next, suppose that \( j = 1 \). In this case we obtain
\[
(4) \quad (L \cdot U)_{i,1} = \sum_{k=1}^{n} l_{i,k} u_{k,1} = l_{1,1} u_{1,1} = 1.
\]
Finally, we assume that $2 \leq i, j \leq n$. We split the proof into two cases, according to the following possibilities for $j$.

**Case 1.** $j \equiv 0$. In this case we claim that

$$
(L \cdot U)_{i,j} = (L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j}.
$$

Since $j - 1 \equiv 1$, by (2) we get

$$
(L \cdot U)_{i-1,j-1} = \sum_{k=1}^{n} l_{i-1,k} u_{k,j-1} = 1 - \sum_{k=1}^{\frac{j-2}{2}} l_{i-1,2k} + 3 \sum_{k=1}^{\frac{j-2}{2}} l_{i-1,2k+1},
$$

and since $j \equiv 0$ we obtain

$$
(L \cdot U)_{i-1,j} = \sum_{k=1}^{n} l_{i-1,k} u_{k,j} = 1 + \sum_{k=1}^{\frac{j-1}{2}} l_{i-1,2k} - \sum_{k=1}^{\frac{j-2}{2}} l_{i-1,2k+1},
$$

and ultimately

$$
(L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j} = 2 + 2 \sum_{k=1}^{\frac{j-2}{2}} l_{i-1,2k+1} + l_{i-1,j}.
$$

Again, since $j \equiv 0$ we obtain

$$
(L \cdot U)_{i,j} = \sum_{k=1}^{n} l_{i,k} u_{k,j} = l_{i,1} + \sum_{k=1}^{\frac{j-1}{2}} l_{i,2k} - \sum_{k=1}^{\frac{j-1}{2}} l_{i,2k+1},
$$

and by (1) we get

$$
(L \cdot U)_{i,j} = l_{i,1} + \sum_{k=1}^{\frac{j-1}{2}} (l_{i-1,2k} + l_{i-1,2k+1}) - \sum_{k=1}^{\frac{j-2}{2}} (l_{i-1,2k} - l_{i-1,2k+1}),
$$

and after some further simplification we obtain

$$
(L \cdot U)_{i,j} = 2 + 2 \sum_{k=1}^{\frac{j-2}{2}} l_{i-1,2k+1} + l_{i-1,j}.
$$

Now, from (6) and (7) we obtain (5).

**Case 2.** $j \equiv 1$. In this case we claim that

$$
(L \cdot U)_{i,j} = (L \cdot U)_{i-1,j-1} - (L \cdot U)_{i-1,j}.
$$

Here, since $j - 1 \equiv 0$, by (2) we obtain

$$
(L \cdot U)_{i-1,j-1} = \sum_{k=1}^{n} l_{i-1,k} u_{k,j-1} = 1 + \sum_{k=1}^{\frac{j-1}{2}} l_{i-1,2k} - \sum_{k=1}^{\frac{j-3}{2}} l_{i-1,2k+1}
$$

and similarly we deduce that

\[(L \cdot U)_{i-1,j} = \sum_{k=1}^{n} l_{i-1,k} u_{k,j} = -\sum_{k=1}^{i-1} l_{i-1,2k} + 3 \sum_{k=1}^{i-1} l_{i-1,2k+1},\]

because \(j \equiv 1\). Therefore by an easy calculation we conclude that

\[(9) \ (L \cdot U)_{i-1,j-1} - (L \cdot U)_{i-1,j} = 2 \sum_{k=1}^{i-1} l_{i-1,2k} - 4 \sum_{k=1}^{i-3} l_{i-1,2k+1} - 3l_{i-1,j}.\]

Again, since \(j \equiv 1\) we have

\[(L \cdot U)_{i,j} = \sum_{k=1}^{n} l_{i,k} u_{k,j} = l_{i,1} - \sum_{k=1}^{i-1} l_{i,2k} + 3 \sum_{k=1}^{i-1} l_{i,2k+1}.\]

Now, by (1) we obtain

\[(L \cdot U)_{i,j} = 1 - \sum_{k=1}^{i-1} (l_{i-1,2k-1} + l_{i-1,2k}) + 3 \sum_{k=1}^{i-3} (l_{i-1,2k} - l_{i-1,2k+1}),\]

and simply we can observe that

\[(10) \ (L \cdot U)_{i,j} = 2 \sum_{k=1}^{i-1} l_{i-1,2k} - 4 \sum_{k=1}^{i-3} l_{i-1,2k+1} - 3l_{i-1,j}.\]

Now, from (9) and (10) we obtain (8).

Therefore, from (3), (4), (5) and (8) we conclude the theorem. \(\square\)

Next, we focus on the sequence \((\det A^{(2, \pm)}\alpha(n))\) for \(n \in \mathbb{N}\).

**Theorem 2.** The sequence \((\det A^{(2, \pm)}\alpha(n))\) of determinants associated to the sequence \(\alpha = (1, 1, 1, \ldots)\) satisfies the following

\[\det A^{(2, \pm)}\alpha(n) = \left\{ \begin{array}{ll} (-5)^{2l-1} & \text{if } n = 4l, (l = 1, 2, \ldots) \\ (-5)^{2l} & \text{if } n = 4l + r, (r = 1, 2, 3, l = 0, 1, 2, \ldots) \end{array} \right.\]

**Proof.** Again, we use the LU-factorization method. Here, we claim that

\[A^{(2, \pm)}\alpha\beta(n) = L \cdot U,\]

where \(L = A^{(2, \pm)}\beta\alpha\beta(n)\) with \(\beta = (1, 0, 0, \ldots)\) and where

\[U = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix},\]
Again, it is immediately obvious that the claimed factorization of $A^{(2, \pm)}_{\alpha, \alpha}(n)$ implies the validity of the theorem.
Suppose that
\[ L = (l_{i,j})_{1 \leq i,j \leq n} \quad \text{and} \quad U = (u_{i,j})_{1 \leq i,j \leq n}. \]

Then by definition, we have \( l_{1,1} = 1, l_{1,j} = 0, l_{i,1} = 1 \) for \( 2 \leq i,j \leq n \) and
\[
(11) \quad l_{i,j} = \begin{cases} 
    l_{i-1,j-1} + l_{i-1,j} & \text{if } j \equiv 4 \mod 2, 3 \\
    l_{i-1,j-1} - l_{i-1,j} & \text{if } j \equiv 0, 1
    \end{cases}
\]
for \( 2 \leq i,j \leq n \). Moreover, the \( j \)th column of \( U \) can be considered as follows:
\[
(12) \quad (u_{1,j}, \ldots, u_{n,j})^T = \begin{cases} 
    (1,0,\ldots,0)^T & j = 1, \\
    (1,-1,3,-5,-1,-1,3,-5,\ldots,-5,0,\ldots,0)^T & j \equiv 0, \text{ \( j \) times (4-periodic)}, \\
    (1,-1,1,1,-5,1,1,1,-5,\ldots,1,0,\ldots,0)^T & j \equiv 1, \text{ \( j \) times (4-periodic)}, \\
    (1,1,1,-1,1,1,-1,1,\ldots,1,1,0,\ldots,0)^T & j \equiv 2, \text{ \( j \) times (4-periodic)}, \\
    (1,1,1,-1,1,1,-1,1,\ldots,1,1,0,\ldots,0)^T & j \equiv 3, \text{ \( j \) times (4-periodic)}
    \end{cases}
\]

For the proof of the claimed factorization we need again some calculations. In fact, the \( (i,j) \)-entry of \( L \cdot U \) is
\[
(L \cdot U)_{i,j} = \sum_{k=1}^{n} l_{i,k} u_{k,j}.
\]

It is easy to see that it is enough to show that \((L \cdot U)_{1,j} = 1, (L \cdot U)_{i,1} = 1\) for \( 1 \leq i,j \leq n \) and
\[
(11) \quad (L \cdot U)_{i,j} = \begin{cases} 
    (L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j} & j \equiv 2, 3 \\
    (L \cdot U)_{i-1,j-1} - (L \cdot U)_{i-1,j} & j \equiv 0, 1
    \end{cases}
\]
for \( 2 \leq i,j \leq n \), in order to prove the theorem.

Again, we verify the claim by a direct calculation. First, suppose that \( i = 1 \). Then
\[
(13) \quad (L \cdot U)_{1,j} = \sum_{k=1}^{n} l_{1,k} u_{k,j} = l_{1,1} u_{1,j} = 1.
\]

Next, suppose that \( j = 1 \). In this case we obtain
\[
(14) \quad (L \cdot U)_{i,1} = \sum_{k=1}^{n} l_{i,k} u_{k,1} = l_{i,1} u_{1,1} = 1.
\]

Finally, we assume that \( 2 \leq i,j \leq n \). We split the proof into four cases, according to the following possibilities for \( j \).
Case 1. \( j \not\equiv 0 \). In this case we claim that

\[(L \cdot U)_{i,j} = (L \cdot U)_{i-1,j-1} - (L \cdot U)_{i-1,j}.\]

Since \( j - 1 \not\equiv 3 \), we obtain

\[(L \cdot U)_{i-1,j-1} = l_{i-1,1} - \sum_{k=1}^{i-4} l_{i-1,4k} - \sum_{k=1}^{i-4} l_{i-1,4k+1} + \sum_{k=0}^{i-4} l_{i-1,4k+2} + \sum_{k=0}^{i-4} l_{i-1,4k+3},\]

and since \( j \not\equiv 0 \), it follows that

\[(L \cdot U)_{i-1,j} = l_{i-1,1} - 5 \sum_{k=1}^{i-4} l_{i-1,4k} - \sum_{k=1}^{i-4} l_{i-1,4k+1} + \sum_{k=0}^{i-4} l_{i-1,4k+2} + 3 \sum_{k=0}^{i-4} l_{i-1,4k+3}.\]

Consequently, we obtain

\[(L \cdot U)_{i-1,j-1} - (L \cdot U)_{i-1,j} = 4 \sum_{k=1}^{i} l_{i-1,4k} + 2 \sum_{k=0}^{i} l_{i-1,4k+2} - 2 \sum_{k=0}^{i} l_{i-1,4k+3} - 5l_{i-1,j}.\]

On the other hand since \( j \equiv 0 \), we get

\[(L \cdot U)_{i,j} = l_{i,1} - 5 \sum_{k=1}^{i} l_{i,4k} - \sum_{k=1}^{i} l_{i,4k+1} - \sum_{k=0}^{i} l_{i,4k+2} + 3 \sum_{k=0}^{i} l_{i,4k+3}.\]

Now by (11) we deduce that

\[(L \cdot U)_{i,j} = 1 - \frac{i}{5} \sum_{k=1}^{i} (l_{i,4k} - l_{i-1,4k}) - \sum_{k=1}^{i} (l_{i-1,4k} - l_{i-1,4k+1})\]

\[\quad - \sum_{k=0}^{i} (l_{i-1,4k+1} + l_{i-1,4k+2}) + 3 \sum_{k=0}^{i} (l_{i-1,4k+2} + l_{i-1,4k+3}),\]

and after some further simplifications the expression reduces to

\[(L \cdot U)_{i,j} = 4 \sum_{k=1}^{i} l_{i-1,4k} + 2 \sum_{k=0}^{i} l_{i-1,4k+2} - 2 \sum_{k=0}^{i} l_{i-1,4k+3} - 5l_{i-1,j}.\]

Now, from (16) and (17) we obtain (15).

Case 2. \( j \equiv 1 \). Here, we claim that

\[(L \cdot U)_{i,j} = (L \cdot U)_{i-1,j-1} - (L \cdot U)_{i-1,j}.\]

Since \( j - 1 \not\equiv 0 \), we obtain

\[(L \cdot U)_{i-1,j-1} = l_{i-1,1} - 5 \sum_{k=1}^{i} l_{i-1,4k} - \sum_{k=1}^{i} l_{i-1,4k+1} - \sum_{k=0}^{i} l_{i-1,4k+2} + 3 \sum_{k=0}^{i} l_{i-1,4k+3}.\]
Similarly, since $j \equiv 1$ it follows that

$$(L \cdot U)_{i-1,j} = l_{i-1,1} + \sum_{k=1}^{i-1} l_{i-1,4k} - 5 \sum_{k=1}^{i-4} l_{i-1,4k+1} - \sum_{k=0}^{i-4} l_{i-1,4k+2} - \sum_{k=0}^{i-5} l_{i-1,4k+3}.$$  

Therefore, we have

(19)

$$(L \cdot U)_{i-1,j-1} - (L \cdot U)_{i-1,j} = -6 \sum_{k=1}^{i-1} l_{i-1,4k} + 4 \sum_{k=1}^{i-4} l_{i-1,4k+1} + 4 \sum_{k=0}^{i-5} l_{i-1,4k+3} - 5l_{i-1,j}.$$  

Furthermore, since $j \equiv 1$ we obtain

$$
(L \cdot U)_{i,j} = l_{i,1} + \sum_{k=1}^{i-1} l_{i,4k} - 5 \sum_{k=1}^{i-4} l_{i,4k+1} - \sum_{k=0}^{i-4} l_{i,4k+2} + \sum_{k=0}^{i-5} l_{i,4k+3}.
$$

Now we apply (11), to get

$$(L \cdot U)_{i,j} = 1 + \sum_{k=1}^{i-1} (l_{i-1,4k-1} - l_{i-1,4k}) - 5 \sum_{k=1}^{i-4} (l_{i-1,4k} - l_{i-1,4k+1})$$

$$- \sum_{k=0}^{i-4} (l_{i-1,4k+1} + l_{i-1,4k+2}) + \sum_{k=0}^{i-5} (l_{i-1,4k+2} + l_{i-1,4k+3}).$$

After some simplifications this leads to

(20)  

$$(L \cdot U)_{i,j} = -6 \sum_{k=1}^{i-1} l_{i-1,4k} + 4 \sum_{k=1}^{i-4} l_{i-1,4k+1} + 4 \sum_{k=0}^{i-5} l_{i-1,4k+3} - 5l_{i-1,j}.$$  

Through comparing (19) and (20), we can get (18).

Case 3. $j \equiv 2$. In this case we claim that

(21)  

$$(L \cdot U)_{i,j} = (L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j}.$$  

Here since $j - 1 \equiv 1$, by (12) we obtain

$$(L \cdot U)_{i-1,j-1} = 1 + \sum_{k=1}^{i-2} l_{i-1,4k} - 5 \sum_{k=1}^{i-4} l_{i-1,4k+1} - \sum_{k=0}^{i-4} l_{i-1,4k+2} + \sum_{k=0}^{i-5} l_{i-1,4k+3}.$$  

and since $j \equiv 2$ it follows that

$$(L \cdot U)_{i-1,j} = 1 + \sum_{k=1}^{i-2} l_{i-1,4k} + 3 \sum_{k=1}^{i-2} l_{i-1,4k+1} + \sum_{k=0}^{i-4} l_{i-1,4k+2} - \sum_{k=0}^{i-6} l_{i-1,4k+3}.$$
Therefore we have

\[(L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j} = 2 + 2 \sum_{k=1}^{\frac{j-2}{2}} l_{i-1,4k} - 2 \sum_{k=1}^{\frac{j-2}{2}} l_{i-1,4k+1} + l_{i-1,j}.\]

On the other hand, since \(j \equiv 2 \mod 4\) we deduce that

\[(L \cdot U)_{i,j} = 1 + \sum_{k=1}^{\frac{j-2}{2}} l_{i-1,4k} + 3 \sum_{k=1}^{\frac{j-2}{2}} l_{i-1,4k+1} + \sum_{k=0}^{\frac{j-2}{2}} l_{i-1,4k+2} - \sum_{k=0}^{\frac{j-2}{2}} l_{i,4k+3},\]

and by (11) we conclude that

\[(L \cdot U)_{i,j} = 1 + \sum_{k=1}^{\frac{j-2}{2}} (l_{i-1,4k-1} - l_{i-1,4k}) + 3 \sum_{k=1}^{\frac{j-2}{2}} (l_{i-1,4k} - l_{i-1,4k+1})
+ \sum_{k=0}^{\frac{j-2}{2}} (l_{i-1,4k+1} + l_{i-1,4k+2}) - \sum_{k=0}^{\frac{j-2}{2}} (l_{i-1,4k+2} + l_{i-1,4k+3}).\]

Now, an easy calculation shows that

\[(L \cdot U)_{i,j} = 2 + 2 \sum_{k=1}^{\frac{j-2}{2}} l_{i-1,4k} - 2 \sum_{k=1}^{\frac{j-2}{2}} l_{i-1,4k+1} + l_{i-1,j}.\]

By comparing (22) and (23), we may obtain (21).

Case 4. \(j \equiv 3 \mod 4\). In this case, we claim that

\[(L \cdot U)_{i,j} = (L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j}.\]

Since \(j - 1 \equiv 2 \mod 4\), we obtain

\[(L \cdot U)_{i-1,j-1} = 1 + \sum_{k=1}^{\frac{j-3}{2}} l_{i-1,4k} + 3 \sum_{k=1}^{\frac{j-3}{2}} l_{i-1,4k+1} + \sum_{k=0}^{\frac{j-3}{2}} l_{i-1,4k+2} - \sum_{k=0}^{\frac{j-3}{2}} l_{i,4k+3},\]

Similarly, since \(j \equiv 3 \mod 4\) it follows that

\[(L \cdot U)_{i-1,j} = 1 - \sum_{k=1}^{\frac{j-3}{2}} l_{i-1,4k} - \sum_{k=1}^{\frac{j-3}{2}} l_{i-1,4k+1} - \sum_{k=0}^{\frac{j-3}{2}} l_{i-1,4k+2} + \sum_{k=0}^{\frac{j-3}{2}} l_{i-1,4k+3},\]

Therefore, we have

\[(L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j} = 2 + 2 \sum_{k=1}^{\frac{j-3}{2}} l_{i-1,4k+1} + 2 \sum_{k=0}^{\frac{j-3}{2}} l_{i-1,4k+2} + l_{i-1,j}.\]
On the other hand, since \( j^4 \equiv 2 \mod 3 \) we obtain

\[
(L \cdot U)_{i,j} = 1 - \sum_{k=1}^{\frac{i-3}{2}} l_{i,4k} - \sum_{k=1}^{\frac{i-3}{2}} l_{i,4k+1} + \sum_{k=0}^{\frac{i-3}{2}} l_{i,4k+2} - \sum_{k=0}^{\frac{i-3}{2}} l_{i,4k+3}.
\]

Again by (11) we conclude that

\[
(L \cdot U)_{i,j} = 1 - \sum_{k=1}^{\frac{i-3}{2}} (l_{i-1,4k-1} - l_{i-1,4k}) - \sum_{k=0}^{\frac{i-3}{2}} (l_{i-1,4k} - l_{i-1,4k+1})
\]

\[
+ \sum_{k=0}^{\frac{i-3}{2}} (l_{i-1,4k+1} + l_{i-1,4k+2}) + \sum_{k=0}^{\frac{i-3}{2}} (l_{i-1,4k+2} + l_{i-1,4k+3}),
\]

and we easily deduce that

\[
(L \cdot U)_{i,j} = 2 + 2 \sum_{k=1}^{\frac{i-3}{2}} l_{i-1,4k+1} + 2 \sum_{k=0}^{\frac{i-3}{2}} l_{i-1,4k+2} + l_{i-1,j}.
\]

By comparing (25) and (26), we can get (24).

Therefore, from (13), (14), (15), (18), (21) and (24) we conclude the theorem. 

In the end, we consider the sequence \((\det A^{(3, \pm)}_{\alpha, \alpha}(n))\) for \(n \in \mathbb{N}\).

**Theorem 3.** The sequence \((\det A^{(3, \pm)}_{\alpha, \alpha}(n))\) of determinants associated to the sequence \(\alpha = (1, 1, 1, \ldots)\) satisfies the following

\[
\det A^{(3, \pm)}_{\alpha, \alpha}(n) = \begin{cases} 
11^{3l-1} & \text{if } n = 6l, \quad (l = 1, 2, \ldots) \\
11^{3l} & \text{if } n = 6l + r, \quad (r = 1, 2, 3, 4, \quad l = 0, 1, 2, \ldots) \\
11^{3l+1} & \text{if } n = 6l + 5. \quad (l = 0, 1, 2, \ldots)
\end{cases}
\]

**Proof.** The proof is similar to the proof of Theorem 1.1 and Theorem 1.2, and we avoid presenting some of the details. Again, we apply LU-factorization. Here, we claim that

\[
A^{(3, \pm)}_{\alpha, \alpha}(n) = L \cdot U,
\]

where \(L = A^{(3, \pm)}_{\beta, \alpha}(n)\) with \(\beta = (1, 0, 0, \ldots)\) is a lower triangular matrix with 1’s on the diagonal, and where

\[
U = \begin{bmatrix}
U_1 \\
U_2 \\
\vdots \\
U_n
\end{bmatrix},
\]
where

\( (u_{i,n-1}, u_{i,n}) \) is satisfied in Table 3.

**Table 3.**

<table>
<thead>
<tr>
<th>( i \mod 6 )</th>
<th>( n \mod 6 = 0 )</th>
<th>( n \mod 6 = 1 )</th>
<th>( n \mod 6 = 2 )</th>
<th>( n \mod 6 = 3 )</th>
<th>( n \mod 6 = 4 )</th>
<th>( n \mod 6 = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i \equiv 0 )</td>
<td>(-1, 1)</td>
<td>(11, -7)</td>
<td>(-7, 1)</td>
<td>(1, 1)</td>
<td>(1, -1)</td>
<td>(-1, -1)</td>
</tr>
<tr>
<td>( i \equiv 1 )</td>
<td>(-1, -1)</td>
<td>(-1, 1)</td>
<td>(11, -5)</td>
<td>(-5, 3)</td>
<td>(3, -1)</td>
<td>(-1, -1)</td>
</tr>
<tr>
<td>( i \equiv 2 )</td>
<td>(-1, -1)</td>
<td>(-1, -1)</td>
<td>(1, 1)</td>
<td>(1, 1)</td>
<td>(1, -1)</td>
<td>(1, -1)</td>
</tr>
<tr>
<td>( i \equiv 3 )</td>
<td>(3, 1)</td>
<td>(1, 1)</td>
<td>(1, -1)</td>
<td>(-1, 1)</td>
<td>(1, 1)</td>
<td>(1, 3)</td>
</tr>
<tr>
<td>( i \equiv 4 )</td>
<td>(-5, 1)</td>
<td>(1, -1)</td>
<td>(-1, 1)</td>
<td>(1, -1)</td>
<td>(-1, 1)</td>
<td>(1, 5)</td>
</tr>
<tr>
<td>( i \equiv 5 )</td>
<td>(11, -7)</td>
<td>(-7, 3)</td>
<td>(3, -1)</td>
<td>(-1, 1)</td>
<td>(1, 1)</td>
<td>(-1, 11)</td>
</tr>
</tbody>
</table>

The matrix \( U \) is an upper triangular one with diagonal entries

\[
1, 1, 1, 1, 11, 11, 11, \ldots, u_{n-1,n-1}, u_{n,n},
\]

6-periodic

where

\[
(u_{n-1,n-1}, u_{n,n}) = \begin{cases} 
(11, 11) & \text{if } n \equiv 0 \text{ or } 1, \\
(11, 1) & \text{if } n \equiv 2, \\
(1, 1) & \text{if } n \equiv 3 \text{ or } 4, \\
(1, 11) & \text{if } n \equiv 5.
\end{cases}
\]
Since \( \det L = 1 \) and
\[
\det U = \begin{cases} 
11^{3l-1} & \text{if } n = 6l, \ (l = 1, 2, \ldots) \\
11^{3l} & \text{if } n = 6l + r, \ (r = 1, 2, 3, 4, \ l = 0, 1, 2, \ldots) \\
11^{3l+1} & \text{if } n = 6l + 5, \ (l = 0, 1, 2, \ldots)
\end{cases}
\]
it is obvious that the claimed factorization of \( A_{n, \alpha}^{(3, \pm)}(n) \) implies the validity of the theorem.

Let us do the required calculation. Again, we assume that
\[
L = (l_{i,j})_{1 \leq i, j \leq n} \quad \text{and} \quad U = (u_{i,j})_{1 \leq i, j \leq n}.
\]
Then by definition, we have \( l_{1,1} = 1, \ l_{1,j} = 0, \ l_{i,1} = 1 \) for \( 2 \leq i, j \leq n \) and the entries \( l_{i,j} \) for \( 2 \leq i, j \leq n \) satisfy
\begin{equation}
(27) \quad l_{i,j} = \begin{cases} 
l_{i-1,j-1} + l_{i-1,j} & \text{if } j \equiv 2, 3, 4 \\
l_{i-1,j-1} - l_{i-1,j} & \text{if } j \equiv 5, 0, 1.
\end{cases}
\end{equation}
Moreover, the \( j \)th column of \( U \) can be considered as follows.
\begin{equation}
(28) \quad (u_{1,j}, \ldots, u_{n,j})^T = \begin{cases} 
(1, 0, \ldots, 0)^T & j = 1, \\
(1, -1, 1, 1, -7, 11, -1, \ldots, -7, 11, 0, \ldots, 0)^T & j \equiv 0, \\
(1, -1, 1, 1, -7, 11, -1, \ldots, -7, 11, 0, \ldots, 0)^T & j \equiv 6, \\
(1, -1, 1, 1, -1, -5, \ldots, -5, 1, 0, \ldots, 0)^T & j \equiv 2, \\
(1, 1, -1, 1, 1, 3, \ldots, 3, 1, 1, 0, \ldots, 0)^T & j \equiv 3, \\
(1, 1, -1, 1, -1, -1, \ldots, -1, 1, 1, 0, \ldots, 0)^T & j \equiv 4, \\
(1, -1, 3, -5, 11, -1, \ldots, -5, 11, 0, \ldots, 0)^T & j \equiv 5.
\end{cases}
\end{equation}

In order to prove the claim we show that the \((i, j)\)-entry of \( L \cdot U \), that is
\[
(L \cdot U)_{i,j} = \sum_{k=1}^{n} l_{i,k} u_{k,j},
\]
satisfy \((L \cdot U)_{1,1} = 1, \ (L \cdot U)_{i,1} = 1 \) for \( 1 \leq i, j \leq n \) and
\begin{equation}
(29) \quad (L \cdot U)_{i,j} = \begin{cases} 
(L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j} & j \equiv 2, 3, 4 \\
(L \cdot U)_{i-1,j-1} - (L \cdot U)_{i-1,j} & j \equiv 5, 0, 1.
\end{cases}
\end{equation}
for \( 2 \leq i, j \leq n \).
First assume that \( i = 1 \). Then, in accordance with the definition of \( l_{1,j} \), we obtain
\[
(L \cdot U)_{1,j} = \sum_{k=1}^{n} l_{1,k} u_{k,j} = l_{1,1} u_{1,j} = 1.
\]

Next, suppose that \( j = 1 \). In this case by (28) we obtain
\[
(L \cdot U)_{i,1} = \sum_{k=1}^{n} l_{i,k} u_{k,1} = l_{i,1} u_{1,1} = 1.
\]

Finally, we assume that \( 2 \leq i, j \leq n \). In this case we must show that the entries \( (L \cdot U)_{i,j} \) satisfy (29). Here, there are six cases to distinguish, according to \( j \equiv 0, 1, 2, 3, 4 \) or \( 5 \). Using similar arguments to those in the proof of Theorem 1.2, we see that the result is true in any cases. For instance, we assume that \( j \equiv 4 \). In this case, we must establish that
\[
(L \cdot U)_{i,j} = (L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j}.
\]

Since \( j \equiv 4 \), in accordance with (28) , the right-hand of (30) is equal to
\[
2 + 2 \sum_{k=1}^{i-4} l_{i-1,6k+1} + 2 \sum_{k=0}^{i-4} l_{i-1,6k+2} + 2 \sum_{k=0}^{i-4} l_{i-1,6k+3} + l_{i-1,j}.
\]

Again, since \( j \equiv 4 \) by (24), we see that the left-hand of (30) is equal to
\[
1 - \sum_{k=1}^{i-4} l_{i,6k} - \sum_{k=1}^{i-4} l_{i,6k+1} - \sum_{k=0}^{i-4} l_{i,6k+2} + \sum_{k=0}^{i-4} l_{i,6k+3} + \sum_{k=0}^{i-4} l_{i,6k+4} - \sum_{k=0}^{i-10} l_{i,6k+5}.
\]

Now, if we substitute the corresponding value for \( l_{i,6k+r} (0 \leq r \leq 5) \) from (29), we can conclude
\[
(L \cdot U)_{i,j} = 2 + 2 \sum_{k=1}^{i-4} l_{i-1,6k+1} + 2 \sum_{k=0}^{i-4} l_{i-1,6k+2} + 2 \sum_{k=0}^{i-4} l_{i-1,6k+3} + l_{i-1,j}
\]
which results in (30). In this way the proof is completed. \( \square \)

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