ABELIAN-BY-NILPOTENT GROUPS WITH CHAIN CONDITIONS FOR NORMAL SUBGROUPS OF INFINITE ORDER OR INDEX

Dae Hyun Paek

Reprinted from the Bulletin of the Korean Mathematical Society
Vol. 44, No. 4, November 2007

©2007 The Korean Mathematical Society
ABELIAN-BY-NILPOTENT GROUPS WITH CHAIN CONDITIONS FOR NORMAL SUBGROUPS OF INFINITE ORDER OR INDEX

Dae Hyun Paek

ABSTRACT. We study the structure of abelian-by-nilpotent groups satisfying the maximal condition on infinite normal subgroups or the minimal condition on normal subgroups of infinite index.

1. Introduction

A group $G$ is said to satisfy the weak maximal condition on normal subgroups if there are no infinite ascending chains $G_1 < G_2 < \cdots$ of normal subgroups of $G$ such that all the indices $|G_{i+1}:G_i|$ are infinite. The weak minimal condition on normal subgroups is defined by substituting descending for ascending chains. Kurdachenko [2] considered groups satisfying the weak maximal or weak minimal conditions on normal subgroups.

A group $G$ is said to satisfy $\text{max-}\infty n$ (the maximal condition on infinite normal subgroups) if there are no infinite ascending chains of infinite normal subgroups of $G$. Similarly a group $G$ is said to satisfy $\text{min-}\infty n$ (the minimal condition on normal subgroups of infinite index) if there are no infinite descending chains of normal subgroups with infinite index in $G$. Since the chain conditions $\text{max-}\infty n$ and $\text{min-}\infty n$ are weaker than the chain conditions $\text{max-}n$ and $\text{min-}n$ (the maximal and minimal conditions on normal subgroups, respectively), we define a group satisfies $\text{max-}\infty n^*$ if it satisfies $\text{max-}\infty n$, but not $\text{max-}n$ and a group satisfies $\text{min-}\infty n^*$ if it satisfies $\text{min-}\infty n$, but not $\text{min-}n$.

De Giovanni et al. [3] characterized the structure of groups satisfying $\text{max-}\infty n^*$ or $\text{min-}\infty n^*$. In addition, the structure of nonfinitely generated solvable groups satisfying $\text{max-}\infty n^*$ and solvable groups satisfying $\text{min-}\infty n^*$ was investigated in detail. In this paper, we consider abelian-by-nilpotent groups with these chain conditions.

Received February 7, 2007.
2000 Mathematics Subject Classification. 20E15.
Key words and phrases. maximal condition, minimal condition.
This work was supported by the Research Center for Elementary Education, Busan National University of Education.
2. Basic results

We say that a $G$-operator group $N$ is $G$-quasifinite if $N$ is infinite but every proper $G$-invariant subgroup of $N$ is finite.

**Lemma 2.1** ([3], Theorem 2.1). Let $G$ be a group satisfying $\max-\infty^n$. Then there is an infinite normal subgroup $R$ which has the following properties:

1. $R$ is the unique smallest normal subgroup such that $G/R$ has $\max-n$;
2. $R$ is the unique $G$-quasifinite normal subgroup;
3. $R$ is either an elementary abelian $p$-group or a radicable abelian $p$-group of finite rank for some prime $p$;
4. $R$ is a countably infinite, locally finite $G$-module.

And the subgroup $R$ in Lemma 2.1 is denoted by $\rho(G)$.

**Lemma 2.2** ([3], Proposition 4.4). If $G$ is a nonfinitely generated soluble group with $\max-\infty^n$, then $C_G(\rho(G))$ is a torsion group.

Karbe [1] proved that the weak maximal or weak minimal conditions on normal subgroups are inherited by any subgroups of finite index. We aim to extend Wilson’s theorem on groups with $\min-n$ to $G$-operator groups. This will be used for investigating abelian-by-nilpotent groups with $\min-\infty^n$. Recall the statement of Wilson’s theorem: if a group $G$ satisfies $\min-n$ and $H$ is a subgroup of $G$ with finite index, then $H$ satisfies $\min-n$.

The following is the generalization of Wilson’s theorem.

**Proposition 2.3.** Let $M$ be a $G$-operator group and let $H$ be a subgroup of $G$ of finite index. If $M$ has $\min-G$, then it has $\min-H$.

**Proof.** First note that the case $M = G$ is Wilson’s Theorem. The proof is substantially Wilson’s. Suppose that $M$ does not in fact have $\min-H$. Since $G/H_G$ is finite, we may assume that $H \triangleleft G$. By $\min-G$ it follows that $M$ contains a subgroup $K$ which is $G$-invariant and minimal with respect to not satisfying $\min-H$.

Consider the set $\mathcal{S}$ of all finite nonempty subsets $X$ of $G$ with the following property: if

\[(1) \quad K_1 > K_2 > \cdots\]

is an infinite descending chain of $H$-invariant subgroups of $K$, then

\[(2) \quad K = K_i^X\]

holds for all $i$. It is not evident that such subsets exist, so our first concern is to produce one.

Let $T$ be a transversal to $H$ in $G$; thus $G = HT$. For any chain the above type we have $K_i^T = K_i^{HT} = K_i^G \leq K$ since $K_i$ is $H$-invariant and $K$ is $G$-invariant. If $K_i^T \neq K$, then $K_i^T$ has the property $\min-H$ by minimality of
Suppose that $K_j = K_{j+1}$ for some $j \geq i$. By this contradiction $K_i^T = K$ for all $i$ and $T \in \mathfrak{S}$.

We now select a minimal element of $\mathfrak{S}$, say $X$. If $x \in X$, then $Xx^{-1} \in \mathfrak{S}$ because $K$ is $G$-invariant. Of course $Xx^{-1}$ is also minimal in $\mathfrak{S}$ and it contains $1$. Thus we may assume that $1 \in X$. If in fact $X$ contains no other element, then (1) and (2) are inconsistent, so that $K$ has min-$H$. Consequently the set

$$Y = X \setminus \{1\}$$

is nonempty. Therefore $Y$ does not belong to $\mathfrak{S}$ by minimality of $X$.

It follows that there exists an infinite descending chain $K_1 > K_2 > \cdots$ of $H$-invariant subgroups of $K$ such that $K_j^Y \neq K$ for some $j$. Define

$$L_i = K_i \cap K_i^Y.$$

Then $L_i$ is a $H$-invariant subgroup of $K$. Also $L_i \supseteq L_{i+1}$. Suppose that $L_i = L_{i+1}$; since $X \in \mathfrak{S}$, we must have $K = K_i^{X+i}$ and

$$K_i = K_i \cap K_i^{X+i} = K_i \cap (K_i+1 \cap K_i^{Y+i}) \subseteq K_i+1 \cap L_i = K_i+1,$$

contradicting $K_i > K_i+1$. Hence $L_i > L_{i+1}$ for all $i$. Therefore $L_i^X = K$ for all $i$, which shows that

$$K_j = K_j \cap L_j^X = K_j \cap (L_j L_j^Y) \subseteq L_j(K_j \cap K_j^Y) = L_j.$$

Hence $K_j = L_j$. Finally, by definition of $L_j$ we obtain $K_j^Y = K_j^X = K$, a contradiction. \hfill \Box

3. Abelian-by-nilpotent groups with max-$\infty$ or min-$\infty$

Our first result describes abelian-by-nilpotent groups with max-$\infty$.

**Theorem 3.1.** An abelian-by-nilpotent group $G$ satisfies max-$\infty$ if and only if there is an infinite abelian normal subgroup $R$ such that $G/R$ is finitely generated, $R$ is $G$-quasifinite, and $C_G(R)$ is torsion.

**Proof.** Suppose that $G$ satisfies max-$\infty$. Then, since finitely generated abelian-by-nilpotent groups satisfy max-$n$, $G$ is not finitely generated. Hence the result follows from Lemmas 2.1 and 2.2.

Conversely, suppose that $G$ has the structure indicated, but does not satisfy max-$\infty$. Let $G_1 < G_2 < \cdots$ be an infinite ascending chain of infinite normal subgroups of $G$.

- **Case:** $G_i \cap R$ is infinite for some $i$. Since $R$ is $G$-quasifinite, $G_i \cap R = R$ and so $R \leq G_i$. Hence $G/R$ does not have max-$n$, a contradiction.

- **Case:** $G_i \cap R$ is finite for all $i$. Since $G_i R/R \simeq G_i/G_i \cap R$ is infinite, $G_i R/R$ is not torsion. Hence $G_i R/R$ has an element $xR$ of infinite order. Since $(x) R \leq G_i R$, we can assume that $x \in G_i$. If $[R, x^2] = 1$, then $x^2 \in C_G(R)$, a contradiction. Hence $[R, x^2]$ is finite with bounded order. Since $[R, x^2]^k = [R, x^2]$, it follows that $[R, x^2, x^k] = 1$ for some $k > 0$ and all
Proposition 3.2. A torsion abelian-by-nilpotent group $G$ with max-$\infty n^*$ is Chernikov.

Proof. Let $R = r(G)$. Then $G/R$ is finitely generated solvable torsion, so it is finite. Write $G = XR$ where $X$ is finite abelian-by-nilpotent. We will show that $R$ is not an elementary abelian $p$-group. Let

$$U = R^p (X \cap R)$$

for some prime $p$. We pass to the group

$$\overline{G} = G/U = (XU/U) \rtimes (R/U).$$

Thus we can assume that $G = X \rtimes R$ with $R$ an elementary abelian $p$-group and $X$ a finite nilpotent group. Write $X = P \times Q$ where $P$ is the $p$-component and $Q$ is the $p'$-component. Write

$$C = C_R(P).$$

Then $C$ is a $\mathbb{Z}_p Q$-module. By Maschke’s Theorem, $C$ is completely reducible, that is, a direct sum of simple submodules—the latter are $X$-invariant and so are normal in $G$. Thus $C$ is a direct summand of finitely many simple submodules; hence $C$ is finite.

Next observe that $PR$ is nilpotent since $P$ is a finite $p$-group and $R$ is an elementary abelian $p$-group. The argument of the last paragraph shows that each $Z_{i+1}(PR)/Z_i(PR)$ is finite. Consequently $PR$ is finite and so is $R$, a contradiction.

Therefore $R$ is a radicable abelian $p$-group of finite rank for some prime $p$ by Lemma 2.1. Hence $G$ is Chernikov. □

We now consider abelian-by-nilpotent groups with min-$\infty n^*$; in order to do this, we begin with polycyclic groups with min-$\infty n^*$.

Lemma 3.3. A polycyclic group $G$ satisfies min-$\infty n^*$ if and only if it is a finite extension of a $G$-rationally irreducible free abelian subgroup of finite rank.

Proof. Suppose that $G$ satisfies min-$\infty n^*$. Let $A$ be a non-trivial free abelian normal subgroup of $G$. Then $G/A$ must be finite since otherwise $A$ has min-$G$. Now let $B$ be a non-trivial $G$-invariant subgroup of $A$. Then $G/B$ is finite, hence so is $A/B$, by the preceding argument. Consequently $A$ is $G$-rationally irreducible.

Conversely, suppose that $A$ is a $G$-rationally irreducible free abelian subgroup of finite rank such that $G/A$ is finite. Suppose that $G$ does not satisfy min-$\infty n$ and let $G_1 > G_2 > \cdots$ be an infinite descending chain of normal subgroups of $G$ with infinite index.
Case: \( G_i \cap A \) is finite for some \( i \). Since \( A \) is torsion-free, \( G_i \cap A = 1 \). Hence \( G_i A/A \cong G_i / G_i \cap A \cong G_i \) is finite, a contradiction.

Case: \( G_i \cap A \) is infinite for all \( i \). Since \( A / A \cap G_i \cong A G_i / G_i \) is finite, and so is \( G / G_i \), a contradiction. Thus \( G \) satisfies \( \min-infty \). Finally, \( G \) does not satisfy \( \min-n \) for \( A \) does not have \( \min-G \). \( \square \)

Now we can determine the structure of abelian-by-nilpotent groups with \( \min-infty^* \).

**Theorem 3.4.** An abelian-by-nilpotent group \( G \) satisfies \( \min-infty^* \) if and only if it has an infinite abelian normal subgroup \( A \) such that:

either

1. \( A \) is a \( G \)-rationally irreducible free abelian subgroup of finite rank such that \( G/A \) is finite

or else

2. \( G/A \) is infinite cyclic-by-finite, \( A \) has \( \min-G \), and \( A/[A, x] \) is finite where \( x \) is any element of infinite order in \( G \).

**Proof.** Suppose that \( G \) has \( \min-infty^* \). Let \( A \) be an abelian normal subgroup of \( G \) with \( G/A \) nilpotent. If \( A \) is finite, then \( G/A \) is a nilpotent group with \( \min-infty^* \). Hence it is infinite cyclic-by-finite and so is \( G \) ([5], Lemma 3.1). Hence \( G \) has the structure given in (1). Thus we now assume that \( A \) is infinite.

Case: \( G/A \) is finite. We will show that \( G \) is polycyclic in this case. Most of the work is to show that \( A \) is not torsion. Suppose for now that we have shown this. Let \( x \in A \) have infinite order and put \( B = \langle x \rangle^{G} \). Since \( G/A \) is finite, \( B \) is a finitely generated infinite abelian normal subgroup of \( G \). Hence \( G/B \) is finite since otherwise \( B \) has \( \min-G \). Therefore \( G \) is polycyclic. Thus it will suffice to argue that \( A \) is not torsion. Note that \( A \) does not have \( \min-G \). Assuming that \( A \) is torsion, we know that it is the direct sum of finitely many non-trivial primary components and only one primary component \( A_p \) can be infinite since \( A \) does not satisfy \( \min-G \). Let

\[
A[p] = \{ a \in A \mid a^p = 1 \}.
\]

If \( A[p] \) is finite, then \( A \) has finite rank. Hence it is a direct sum of finitely many cyclic and quasicyclic groups. But then \( A \) has \( \min \), as must \( G \), a contradiction. Therefore \( A[p] \) is infinite elementary abelian.

If \( G/A[p] \) is infinite, then \( A[p] \) has \( \min-G \); and hence has \( \min-A \) by Proposition 2.3. This implies that \( A[p] \) is finite, a contradiction. Hence \( G/A[p] \) is finite and \( A[p] \) does not have \( \min-G \). If \( H \) is an infinite \( G \)-invariant subgroup of \( A[p] \) of infinite index, then as before \( H \) has \( \min-A \) by Proposition 2.3, a contradiction. Consequently every infinite \( G \)-invariant subgroup of \( A[p] \) has finite index. It follows that \( A[p] \) has max-\( \infty \) (the maximal condition for infinite \( G \)-invariant subgroups), so \( G \) has max-\( \infty \). Hence \( G \) is Chernikov by Proposition 3.2 and so \( A[p] \) is finite. By this contradiction \( A \) is not torsion.

Case: \( G/A \) is infinite. We will show that \( G \) has the structure given in (2) in this case. Since \( G/A \) is infinite, \( A \) has \( \min-G \) and so \( G/A \) does not have \( \min-n \).
Thus $G/A$ is cyclic-by-finite ([5], Lemma 3.1), and so it is finite-by-cyclic. Now we write $G = XA$ with $X$ a finitely generated subgroup.

Let $z \in A$ have infinite order. Then, since $A$ has min-$G$, it follows that $\langle z \rangle^G$ has min-$G$. Also $\langle z \rangle^G$ is a finitely generated $G/A$-module. Since $G/A$ is finitely generated nilpotent, it is polycyclic. Hence $\langle z \rangle^G$ has max-$G$ ([7], 15.3.3). It follows that $\langle z \rangle^G$ is finite, a contradiction. Therefore $A$ is torsion.

Now let $M$ be a maximal normal torsion subgroup of $G$ containing $A$ such that $G/M = \langle xM \rangle$ is infinite cyclic and $M/A$ is finite.

Now we write $G/A = M/A \times \langle xA \rangle$ where $|x| = \infty$. We note that

\[ \langle x, [M, x] \rangle \triangleleft \langle x, M \rangle = G. \]

If $M/[M, x]$ is infinite, then $\langle x, [M, x] \rangle$ has infinite index in $G$. Thus $\langle x, [M, x] \rangle$ has min-$G$. But then $\langle x^k, [M, x^k] \rangle$ is a $G$-invariant subgroup of $\langle x, [M, x] \rangle$ for each $k > 0$, which cannot be true. Hence $M/[M, x]$ is finite.

Write $G = XA$ where $X$ is a finitely generated abelian-by-nilpotent group. Then $X$ has max-$n$. Hence $X \cap A$ has max-$X$, and also min-$X$ since $G = XA$. Therefore $X \cap A$ is finite. Now factor out by the finite normal subgroup $X \cap A$. Then

\[ G = X \times A \quad \text{and} \quad M = (M \cap X) \times A. \]

Let $x \in G$ with $|x| = \infty$. Then $x = ya$ with $y \in X$, $a \in A$ and clearly $|y| = \infty$. Also $[A, x] = [A, y]$. So we can assume that $x \in X$. Then $[M, x] = [A, x]$ since $[M, x] \leq [A, x][M \cap X, x]$ and $[M \cap X, x] \leq X \cap A = 1$. This argument shows that

\[ [M, x] \leq [A, x](X \cap A). \]

Since $X \cap A$ is finite, so is $[M, x]/[A, x]$. Therefore $A/[A, x]$ is finite.

Conversely, if (1) holds, then $G$ is polycyclic and the result follows from Lemma 3.3. Thus we assume that (2) holds. Suppose that $G_1 > G_2 > \cdots$ is an infinite descending chain of normal subgroups of $G$ with infinite index.

**Case:** $G_i/A$ is infinite for some $i$. $G_iA/A$ contains an element $xA$ of infinite order where $x \in G_i$ and $G/G_iA$ is finite. Since $[A, x] \leq A \cap G_i$, it follows that $A/A \cap G_i \simeq AG_i/G_iA$ is finite and so is $G/G_iA$, a contradiction.

**Case:** $G_iA/A$ is finite for all $i$. There is an $i$ such that $G_iA = G_{i+1}A$ and $G_i \cap A = G_{i+1} \cap A$, which implies that

\[ G_{i+1} = G_{i+1} \cap G_iA = G_i(G_{i+1} \cap A) = G_i, \]

a contradiction.

Therefore $G$ has min-$\infty n$. Finally if $G$ has min-$n$, then it is locally finite ([6], Theorem 5.25). Hence $G/A$ is finitely generated locally finite and so is finite, a contradiction. □

**Example 3.5** ([4], Example 3.4). Let $M = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots$ be an infinite elementary abelian $p$-group and let $X = \langle x \rangle$ be an infinite cyclic group acting on $M$ via

\[ a_i^x = a_1 \text{ and } a_i^{x+1} = a_{i+1}a_i \]
for all \( i = 1, 2, \ldots \). Then \( G = X \rtimes M \) is an abelian-by-nilpotent group with \( \min\)-\( \infty \nabla \).