AN IDEAL-BASED ZERO-DIVISOR GRAPH OF 2-PRIMAL NEAR-RINGS

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Abstract. In this paper, we give topological properties of collection of prime ideals in 2-primal near-rings. We show that Spec(N), the spectrum of prime ideals, is a compact space, and Max(N), the maximal ideals of N, forms a compact T₁-subspace. We also study the zero-divisor graph Γ_I(R) with respect to the completely semiprime ideal I of N. We show that Γ_P(R), where P is a prime radical of N, is a connected graph with diameter less than or equal to 3. We characterize all cycles in the graph Γ_P(R).

1. Preliminaries

In [3], Beck introduced the concept of a zero-divisor graph of a commutative ring with identity, but this work was mostly concerned with coloring of rings. In [2], Anderson and Livingston associated a graph (simple) Γ(R) to a commutative ring R with identity with vertices Z(R)* = Z(R)\{0}, the set of nonzero zero-divisor of R, and for distinct x, y ∈ Z(R)*, the vertices x, and y are adjacent if and only if xy = 0. They investigated the interplay between the ring-theoretic properties of R and the graph-theoretic properties of Γ(R).

In [9], Redmond has generalized the notion of the zero-divisor graph. For a given ideal I of R, he defined an undirected graph Γ_I(R) with vertices \{x ∈ R\I : xy ∈ I for some y ∈ R\I\}, where distinct vertices x and y are adjacent if and only if xy ∈ I.

In this paper, we study the undirected graph Γ_I(N) of near-rings for any completely semiprime ideal I of N. We extend the results obtained by K. Samei [11] for reduced rings to 2-primal near-rings. Clearly, reduced rings are 2-primal near-rings.

Let N be a near-ring with identity. Let J be a completely semiprime ideal of N. The zero-divisor graph of N with respect to the ideal J, denoted by Γ_J(N), is the graph whose vertices are the set \{x ∈ N\J : xy ∈ J for some y ∈ N\J\} with distinct vertices x and y are adjacent if and only if xy ∈ J. If J = 0, then
\( \Gamma_J(N) = \Gamma(N) \), and \( J \) is a non-zero completely prime ideal of \( N \) if and only if \( \Gamma_J(N) = \phi \).

**Example 1.1.** Let \( N = (F_2, \mathcal{F}) \), where \( F = \{0, 1\} \) is the field under addition and multiplication modulo 2. Then its prime radical \( P = \{(000), (010)\} \) is a completely reflexive ideal of the near-ring \( N \) and its ideal based zero-divisor graph \( \Gamma_P(N) \) is:

\[
\begin{array}{ccc}
(000) & \xleftarrow{1} & (010) \\
(100) & \xrightarrow{1} & (011) \\
(110) & \xrightarrow{1} & (011)
\end{array}
\]

**Remark 1.2.** In the above example, \( N \) is a 2-primal near-ring, but neither reduced nor commutative.

Throughout this paper \( N \) is a zero symmetric near-ring with identity unless otherwise stated, and its prime radical is not a prime ideal of \( N \).

Let \( \mathbb{P} \) denote the prime radical, and let \( N(N) \) denote the set of nilpotent elements of \( N \). For any vertices \( x, y \) in a graph \( G \), if \( x \) and \( y \) are adjacent, we denote it as \( x \sim y \). A near-ring \( N \) is called a 2-primal if \( \mathbb{P} = N(N) \). A near-ring \( N \) is said to be reduced if \( N(N) = 0 \). Clearly, reduced near-rings are 2-primal, but the converse need not be true (See Example 1.3 of [5]). A near-ring \( N \) is called \( pm \) if each prime ideal in \( N \) is contained in a unique maximal ideal of \( N \).

We use \( \text{Spec}(N) \), \( \text{Max}(N) \), and \( \text{Min}(N) \) for the spectrum of prime ideals, maximal ideal and minimal prime ideals of \( N \), respectively.

For any ideal \( J \) of \( N \) and \( a \in N \), we define \( V(a) = \{ P \in \text{Spec}(N) : a \in P \} \) and \( D(J) = \text{Spec}(N) \setminus V(J) \). Let \( V(J) = \bigcap_{a \in J} V(a) \). Then \( F = \{ V(J) : J \text{ is an ideal of } N \} \) is closed under finite union and arbitrary intersections, so that there is a topology on \( \text{Spec}(N) \) for which \( F \) is the family of closed sets. This is called the Zariski topology. Note that \( V(A) = \{(J)\} \) for any subset \( A \) of \( N \). Let \( \mathcal{B} = \{ D(a) : a \in N \} \). Then \( \mathcal{B} \) is a basis for a topology on \( \text{Spec}(N) \).

The operations \( cl \) and \( int \) denote the closure and the interior in \( \text{Spec}(N) \). We also set \( V^+(a) = V(a) \cap \text{Min}(N) \); \( D^+(a) = D(a) \cap \text{Min}(N) \).

For any subset \( S \) of \( N \), we define \( \mathcal{P}_S = \{ n \in N : nS \subseteq \mathbb{P} \} \). We set \( \text{Supp}(a) = \bigcap_{x \in \mathcal{P}, V(x)} \).

For distinct vertices \( x \) and \( y \) of \( \Gamma_P(N) \), let \( d(x, y) \) be the length of the shortest path from \( x \) to \( y \). The diameter of a connected graph is the supremum of the distances between vertices. The associated number \( e(a) \) for a vertex \( a \) in \( \Gamma_P(R) \) is defined by \( e(a) = \max\{d(a, b) : a \neq b\} \).

A graph \( G \) is called triangulated (hyper-triangulated) if each vertex (edge) of \( G \) is a vertex (edge) of a triangle.
A point $P$ of $\text{Spec}(N)$ is said to be quasi-isolated if $P$ is a minimal prime ideal and $P$ is not contained in the union of all minimal prime ideals of $N$ different from $P$.

If $a$ and $b$ are the two vertices in $\Gamma_p(N)$, by $c(a, b)$ we mean the length of the smallest cycle containing $a$ and $b$. For every two vertices $a$ and $b$, all possible cases for $c(a, b)$ are given in Theorem 3.9. In this paper the notations of graph theory are from [4], the notations of near-ring are from [8], and the notations of topology are from [6] and [7].

2. Topological space of $\text{Spec}(N)$

In this section, we associate the near-ring properties of $N$ and the topological properties of $\text{Spec}(N)$. We start this section with the following useful lemma.

**Lemma 2.1.** Let $N$ be a near-ring. If $A$ is a subset of $\text{Spec}(N)$, then there exists an ideal $J = \cap A$ of $N$ with $\text{cl}(A) = V(J)$. In particular, if $A$ is a closed subset of $\text{Spec}(N)$, then $A = V(J)$ for some ideal $J$ of $N$.

**Proof.** Let $P_1 \in V(J)$ and let $D(x)$ be any arbitrary element in $B$ such that $P_1 \in D(x)$. Suppose that $D(x) \cap A = \emptyset$. Then $x \notin J$, and so $P_1 \notin V(x)$, a contradiction. Thus $D(x) \cap A \neq \emptyset$, and hence, the result follows from Theorem 17.5 of [7].

In view of above lemma, we have the following remarks.

**Remark 2.2.** Let $N$ be a near-ring.

(i) The closure of $P \in \text{Spec}(N)$ is $V(P)$.

(ii) A point $P \in \text{Spec}(N)$ is closed if and only if $P \in \text{Max}(N)$.

(iii) If $P, Q \in \text{Spec}(N)$ with $\text{cl}(P) = \text{cl}(Q)$, then $P = Q$.

With the help of Lemma 2.1, we have the following some important characterizations of $\text{Spec}(N)$.

**Theorem 2.3.** Let $N$ be a near-ring.

(i) If $F \subseteq \text{Spec}(N)$ is a closed set and $D(K)$ is an open set in $\text{Spec}(N)$ satisfying $F \cap \text{Max}(N) \subseteq D(K)$, then $F \subseteq D(K)$.

(ii) $\text{Spec}(N)$ is a compact space.

(iii) $\text{Max}(N)$ is a compact $T_1$ subspace.

(iv) If $\text{Spec}(N)$ is normal, then $\text{Max}(N)$ is a Hausdorff space.

(v) If $\mathbb{P} = \cap \text{Max}(N)$ and $\text{Max}(N)$ is a Hausdorff space, then $\text{Spec}(N)$ is normal.

**Proof.** (i) Suppose that there is $P \in F$ with $P \notin D(K)$. Then $K + L \subseteq P$ since $F = V(L)$ for some ideal $L$ of $N$. Hence, each maximal ideal $M$ containing $P$ is also in $F$. Then $M \in F \cap \text{Max}(N)$, and so $M \notin D(K)$, a contradiction.

(ii) Let $B = \{D(s_i) : s_i \in J\}$ be the basis of $N$, for any subset $J$ of $N$, and suppose that $\text{Spec}(N) = \cup_{j \in J}D(s_j)$. Then $\phi = \cap_{j \in J}(\text{Spec}(N) \setminus D(s_j)) = \cap_{j \in J}V(s_j) = V(\cup_{j \in J}(s_j)) = V(\sum_{j \in J}(s_j))$ which gives $\sum_{j \in J}(s_j) = N$. Then
there exists $K \subset J$ finite with $1 = \sum k \in K s'_k$, where $s'_k \in \langle s_k \rangle$ which implies $\text{Spec}(N) = \cup k \in K D(s'_k)$. Indeed, clearly $\cup k \in K D(s'_k) \subseteq \text{Spec}(N)$ and suppose $P \in \text{Spec}(N)$ with $P \not\in \cup k \in K D(s'_k)$. Then $s'_k \not\in P$ for all $k \in K$ which implies $1 \in P$, a contradiction. Hence $\text{Spec}(N)$ is a compact space.

(iii) Let $B = \{ D(s_i) : s_i \in J \}$ be the basis of $N$, for any subset $J$ of $N$, and suppose that $\text{Max}(N) = (\cup j \in J D(s_j)) \cap \text{Max}(N)$. Then

$$\phi = \cap i \in J(\text{Max}(N) \setminus D(s_i)) = (\cap j \in J V(s_j)) \cap \text{Max}(N)$$

$$= V(\sum_{i \in I} (s_i)) \cap \text{Max}(N)$$

which imply $\sum_{i \in J} (s_i) = N$. Then there exists $J_1 \subset J$ finite with $1 = \sum j \in J_1 s_j$, and so $\text{Max}(N) = \cup j \in J_1 D(s_j)$.

Let $M_1$ and $M_2$ be two distinct elements in $\text{Max}(N)$. Then $M_1 \in D(M_2)$ and $M_2 \in D(M_1)$, and so $\text{Max}(N)$ is a $T_1$ space.

(iv) Let $M_1$ and $M_2$ be distinct elements in $\text{Max}(N)$. Then $\{M_1\}$ and $\{M_2\}$ are closed subsets in both $\text{Spec}(N)$ and $\text{Max}(N)$. If $\text{Spec}(N)$ is normal, then there exist disjoint open sets $D(I)$ and $D(J)$ such that $\{M_1\} \subseteq D(I)$ and $\{M_2\} \subseteq D(J)$ for some ideals $I$ and $J$ of $N$, respectively. So, $M_1 \in D(I) \cap \text{Max}(N)$, and $M_2 \in D(J) \cap \text{Max}(N)$, which imply $\text{Max}(N)$ is a Hausdorff space.

(v) Let $F_1$ and $F_2$ be two disjoint closed subsets of $\text{Spec}(N)$. Then $F_1 \cap \text{Max}(N)$ and $F_2 \cap \text{Max}(N)$ are also disjoint subsets of $\text{Max}(N)$. By Theorem 32.3 in [7], $\text{Max}(N)$ is normal. So, there are open subsets $D(J)$ and $D(J_1)$ of $\text{Spec}(N)$ such that $F_1 \cap \text{Max}(N) \subseteq A$, $F_2 \cap \text{Max}(N) \subseteq B$ and $A \cap B = \phi$, where $A = D(J) \cap \text{Max}(N)$ and $B = D(J_1) \cap \text{Max}(N)$.

Assume $\mathbb{P} = \cap \text{Max}(N)$. Then $J \cap \text{Max}(N) = \mathbb{P}$ since $D(J) \cap D(J_1) = D(J \cap J_1)$, and so $D(J) \cap D(J_1) = \phi$. By (i), we have $F_1 \subseteq D(J)$ and $F_2 \subseteq D(J_1)$.

**Theorem 2.4.** Let $N$ be a 2-primal near-ring. Then $\mathbb{P}_S = \cap V(\mathbb{P}_S)$ for any subset $S$ of $N$.

**Proof.** Clearly, $\mathbb{P}_S \subseteq \cap V(\mathbb{P}_S)$. Let $a \in N \setminus \mathbb{P}_S$. Then as $\not\in P$ for some $P \in \text{Spec}(N)$ and $s \in S$ which implies $\mathbb{P}_S \subseteq P$. Thus, $a \not\in P \in V(\mathbb{P}_S)$, and hence, $\cap V(\mathbb{P}_S) \subseteq \mathbb{P}_S$. \(\square\)

**Lemma 2.5.** Let $N$ be a 2-primal near-ring and let $a, b \in N$. Then $\text{int } V(a) \subseteq \text{int } V(b)$ if and only if $\mathbb{P}_a \subseteq \mathbb{P}_b$.

**Proof.** Let $\text{int } V(a) \subseteq \text{int } V(b)$ for any $a, b \in N$ and let $x \in \mathbb{P}_a$. Then $\text{Spec}(N) \setminus V(x) \subseteq \text{int } V(a) \subseteq \text{int } V(b) \subseteq \text{int } V(b)$, which gives $bx \in \mathbb{P}$, so $x \in \mathbb{P}_b$.

Conversely, let $\mathbb{P}_a \subseteq \mathbb{P}_b$ and let $P \in \text{int } V(a)$. Suppose $P \not\in V(b)$. By Lemma 2.1, if $P \not\in \text{Spec}(N) \setminus \text{int } V(a)$, then there is $0 \neq \epsilon \in N$ with $\text{Spec}(N) \setminus \text{int } V(a) \subseteq V(\epsilon)$ and $\epsilon \not\in P$. Clearly $ac \in \mathbb{P}$ and $bc \not\in P$. Then $c \in \mathbb{P}_a$ and $c \not\in \mathbb{P}_b$, a contradiction. \(\square\)
Lemma 2.6. Let $N$ be a 2-primal near-ring. Then for every $a \in N$, $\text{cl}(D(a)) = V(P_a) = \text{Supp}(a) = \text{Spec}(N) \setminus \text{int} V(a)$.

Proof. Let $a \in N$, $P \in V(P_a)$, and let $D(x)$ be any arbitrary basis element in $B$ such that $P \in D(x)$. Let $P \notin D(a)$ and suppose $D(a) \cap D(x) = \phi$. Then $D(xa) \subseteq D(x) \cap D(a) = \phi$, and so $xa \in P$ which implies $x \in P$, a contradiction. Thus, $D(a) \cap D(x) \neq \phi$, and hence, $V(P_a) = \text{cl}(D(a))$.

Let $P \in \text{cl}(D(a))$ and suppose that $P \in \text{int} V(a)$. Then there exists an open set $U$ of $\text{Spec}(N)$ with $P \in U \subseteq V(a)$, and so $P \notin \text{Spec}(N) \setminus U$, a contradiction. Let $P \in \text{Spec}(N) \setminus \text{int} V(a)$ and let $D(x)$ be any arbitrary element in $B$ with $P \in D(x)$. Suppose that $D(x) \cap D(a) = \phi$. Then $P \in D(P_a) \subseteq V(a)$, a contradiction. □

The following result gives the condition under which a subset of $\text{Spec}(N)$ of 2-primal near-ring to be clopen, which will be used in our main result in Section 3.

Lemma 2.7. Let $N$ be a 2-primal near-ring. Then $A$ is a clopen subset of $\text{Spec}(N)$ if and only if there exists an element $a \in N$ with $a \in P$ or $-1 + a \in P$ for all $P \in \text{Spec}(N)$ and $A = V(a)$.

Proof. Suppose that $A$ is a clopen subset of $\text{Spec}(N)$. Let $J = \cap A$ and $J_1 = \cap A^c$. Then by Lemma 2.1 $A = cl(A) = V(J)$ and $A^c = V(J_1)$. So, $V(J) \cap V(J_1) = \phi$, which gives $J + J_1 = N$. Then there exists $a \in J$ and $a' \in J_1$ such that $a + a' = 1$. Therefore $a(-1 + a) \in P$. Thus, for every prime ideal $P$, we have $a \in P$ or $-1 + a \in P$. Consequently, $A = V(J) = V(a)$. The converse is trivial. □

Theorem 2.8. Let $N$ be a 2-primal and pm near-ring. Then $\text{Max}(N)$ is a compact Hausdorff space.

Proof. By Lemma 2.3(iii), $\text{Max}(N)$ is a compact space. Let $M_1, M_2 \in \text{Max}(N)$ and consider the multiplicative subset

$$S = \{a_1b_1 \cdots a_nb_{n-1}b_n : a_i \notin M_1, b_i \notin M_2, \ n, \ i \in \{1, 2, \ldots, n\}\}.$$ 

Suppose that $0 \notin S$. Then there is a prime ideal $P$ of $N$ with $P \cap S = \phi$ and hence $P \subseteq M_1 \cap M_2$, a contradiction. So, there exist $a_i \notin M_1$ and $b_i \notin M_2$ such that $a_1b_1 \cdots a_nb_n = 0$. We now have elements $x_1 \notin M_1$ and $x_2 \notin M_2$ with $x_1x_2 \notin P$, which imply $D(x_1)$ and $D(x_2)$ are disjoint with $M_1 \in D(x_1)$ and $M_2 \in D(x_2)$. □

The following is an immediate corollary of Theorem 2.8.

Corollary 2.9 ([12], Lemma 2.1). If $R$ is a 2-primal and pm ring, then $\text{Max}(R)$ is a compact Hausdorff space.
3. Distance and cycles in $\Gamma_\mathcal{P}(N)$

In this section, we associate the near-ring properties of $N$ and the graph properties of $\Gamma_\mathcal{P}(N)$.

**Theorem 3.1.** Let $N$ be a 2-primal near-ring. Then $\Gamma_\mathcal{P}(N)$ is connected and $\text{diam}(\Gamma_\mathcal{P}(N)) \leq 3$.

**Proof.** Let $x, y \in \Gamma_\mathcal{P}(N)$ be distinct. If $xy \in \mathcal{P}$, then $d(x, y) = 1$. Otherwise, there are $a, b \in N \setminus (\mathcal{P} \cup \{x, y\})$ such that $ax, by \in \mathcal{P}$.

If $a = b$, then $x \approx a \approx y$ is a path of length 2. Thus, we assume that $a \neq b$. If $ab \in \mathcal{P}$, then $x \approx a \approx b \approx y$ is a path of length 3; and hence $d(x, y) \leq 3$. Otherwise, $x \approx ab \approx y$ is a path of length 2; thus, $d(x, y) = 2$. Hence, $d(x, y) \leq 3$. \hfill \Box

**Lemma 3.2.** Let $N$ be a 2-primal near-ring and let $a, b \in \Gamma_\mathcal{P}(N)$. Then

(i) $\text{Supp}(a) \cup \text{Supp}(b) \neq \text{Spec}(N)$ if and only if $\text{Supp}(a) \cup \text{Supp}(b) \subseteq V(c)$ for some $c \in \Gamma_\mathcal{P}(N)$.

(ii) $D(a) \cap D(b) \neq \emptyset$ if and only if there exists $c \in \Gamma_\mathcal{P}(N)$ such that $\emptyset \neq D(a) \cap D(b) \subseteq V(c)$.

**Proof.** (i) Suppose $\text{Supp}(a) \cup \text{Supp}(b) \neq \text{Spec}(N)$. Then there exists an element $P \in \text{Spec}(N)$ with $x, y \notin P$ for some $x \in \mathcal{P}_a$ and $y \in \mathcal{P}_b$. So, $xy \notin \mathcal{P}$. It is easy to see that $\text{Supp}(a) \cup \text{Supp}(b) \subseteq V(xy)$.

Conversely, suppose that $\text{Supp}(a) \cup \text{Supp}(b) = \text{Spec}(N)$. Then $c \in \mathcal{P}$, a contradiction. Hence, $\text{Supp}(a) \cup \text{Supp}(b) \neq \text{Spec}(N)$.

(ii) Straightforward. \hfill \Box

Now by Theorem 3.1, and Lemma 3.2, we have the following characterization of the diameter of $\Gamma_\mathcal{P}(N)$.

**Theorem 3.3.** Let $N$ be a 2-primal near-ring and let $a, b \in \Gamma_\mathcal{P}(N)$ be distinct elements. Then

(i) For any $c \in \Gamma_\mathcal{P}(N)$, we have $c$ is adjacent to both $a$ and $b$ if and only if $\text{Supp}(a) \cup \text{Supp}(b) \subseteq V(c)$.

(ii) $d(a, b) = 1$ if and only if $D(a) \cap D(b) = \emptyset$.

(iii) $d(a, b) = 2$ if and only if $D(a) \cap D(b) \neq \emptyset$ and $\text{Supp}(a) \cup \text{Supp}(b) \neq \text{Spec}(N)$.

(iv) $d(a, b) = 3$ if and only if $D(a) \cap D(b) \neq \emptyset$ and $\text{Supp}(a) \cup \text{Supp}(b) = \text{Spec}(N)$.

**Proof.** (i) Let $c \in \Gamma_\mathcal{P}(N)$. Then $c$ is adjacent to both $a$ and $b$ if and only if $D(a) \cap D(c) = D(b) \cap D(c) = \emptyset$ if and only if $\text{Supp}(a) \cup \text{Supp}(b) \subseteq V(c)$.

(ii) Trivial.

(iii) Let $a, b \in \Gamma_\mathcal{P}(N)$. Then $d(a, b) = 2$ if and only if $ab \notin \mathcal{P}$ and there exists $c \in \Gamma_\mathcal{P}(N)$ such that $c$ is adjacent to both $a$ and $b$ if and only if $D(a) \cap D(b) \neq \emptyset$ and $\text{Supp}(a) \cup \text{Supp}(b) \subseteq V(c)$ if and only if $D(a) \cap D(b) \neq \emptyset$ and $\text{Supp}(a) \cup \text{Supp}(b) \neq \text{Spec}(N)$ by Lemma 3.2.
(iv) By Theorem 3.1, \( d(a, b) = 3 \) if and only if \( d(a, b) \neq 1, 2 \) if and only if \( D(a) \cap D(b) \neq \emptyset \) and \( \text{Supp}(a) \cup \text{Supp}(b) = \text{Spec}(N) \) by (i) and (ii).

Since the reduced commutative ring is also a 2-primal near-ring, the following corollary is immediate.

**Corollary 3.4 ([11], Proposition 2.2).** Let \( R \) be a commutative reduced ring and let \( a, b, c \in \Gamma(R) \) be distinct elements. Then

(i) \( c \) is adjacent to both \( a \) and \( b \) if and only if \( \text{Supp}(a) \cup \text{Supp}(b) \subseteq V(c) \).

(ii) \( d(a, b) = 1 \) if and only if \( D(a) \cap D(b) = \emptyset \).

(iii) \( d(a, b) = 2 \) if and only if \( D(a) \cap D(b) \neq \emptyset \) and \( \text{Supp}(a) \cup \text{Supp}(b) \neq \text{Spec}(R) \).

(iv) \( d(a, b) = 3 \) if and only if \( D(a) \cap D(b) \neq \emptyset \) and \( \text{Supp}(a) \cup \text{Supp}(b) = \text{Spec}(R) \).

The following theorem shows that every minimal prime ideal of 2-primal near-ring that doesn’t contain both \( a \) and \( \mathbb{P}_a \) for any \( a \in N \).

**Theorem 3.5.** Let \( N \) be a 2-primal near-ring and let \( a \in N \). Then \( V'(a) = D'(\mathbb{P}_a) \) and \( D'(a) = V'(\mathbb{P}_a) \). In particular, \( V'(a) \) and \( V'(\mathbb{P}_a) \) are disjoint clopen subsets of \( \text{Spec}(N) \). Also, \( \text{Min}(N) \) is a Hausdorff space.

**Proof.** Let \( P \in V'(a) \) and suppose \( P \notin D'(\mathbb{P}_a) \). Let \( M = \{a, a^2, \ldots\} \) be multiplicative closed system and let \( S = \{I \subseteq N : I \not\subseteq P \text{ and } I \cap M = \emptyset\} \). Since \( \mathbb{P}_a \in S \), \( S \neq \emptyset \). Then by Zorn’s Lemma, there exists a maximal ideal \( \mathcal{P} \) in \( S \) with \( \mathcal{P} \subseteq P \) and \( \mathcal{P} \cap M = \emptyset \). Let \( J \) and \( J_1 \) be ideals of \( N \) such that \( \mathcal{P} \subset J \) and \( \mathcal{P} \subset J_1 \).

Case (i): If \( P \subset J \) and \( P \subset J_1 \), then \( JJ_1 \not\subseteq P \). So \( JJ_1 \not\subseteq \mathcal{P} \).

Case (ii): If \( J \subseteq P \) and \( J_1 \subseteq P \), then \( J \cap M \neq \emptyset \) and \( J_1 \cap M \neq \emptyset \). Then there exist \( j \in J \cap M \) and \( j_1 \in J_1 \cap M \) with \( j' \neq j_1 \in M \) for some \( j' \in J \) and \( j_1' \in J_1 \), which gives \( JJ_1 \cap M \neq \emptyset \). So \( JJ_1 \not\subseteq \mathcal{P} \).

Case (iii): If \( J \subseteq P \) and \( P \subset J_1 \), then by Case (ii), we have \( JP \not\subseteq \mathcal{P} \). So \( JJ_1 \not\subseteq \mathcal{P} \).

Thus, \( \mathcal{P} \) is a prime ideal with \( \mathcal{P} \subset P \), contradicting the minimality of \( P \). Hence, \( V'(a) = D'(\mathbb{P}_a) \). Similarly, we have \( D'(a) = V'(\mathbb{P}_a) \).

Let \( P \neq P' \in \text{Min}(N) \) and \( a \in P \setminus P' \). Then \( V'(a) \) and \( V'(\mathbb{P}_a) \) are disjoint open sets containing \( P \) and \( P' \), respectively.

**Lemma 3.6.** Let \( N \) be a 2-primal near-ring and let \( a \in \Gamma(N) \). If \( e(a) = 1 \), then \( \mathbb{P}_a \) is a completely prime ideal of \( N \).

**Proof.** Straightforward.

**Theorem 3.7.** Let \( N \) be a 2-primal near-ring and \( 2 \notin P \). Then

(i) \( \Gamma(N) \) is a triangulated graph if and only if \( \text{Spec}(N) \) has no quasi-isolated points.
(ii) $\Gamma_P(N)$ is a hyper-triangulated graph if and only if $\text{Spec}(N)$ is connected space and for any $a, b \in \Gamma_P(N)$, we have that $ab \in P$ and $D(a) \cup D(b) \neq \text{Spec}(N)$ imply $\text{Supp}(a) \cup \text{Supp}(b) \neq \text{Spec}(N)$.

(iii) If $2 \notin \Gamma_P(N)$, then every vertex of $\Gamma_P(N)$ is a 4-cycle vertex.

**Proof.** (i) Let $\Gamma_P(N)$ be a triangulated graph and suppose $\text{Spec}(N)$ has a quasi-isolated point $P$. Then $D(P_a) = V(a) = \{P\}$ for some $a \in P$. Clearly, $a \in \Gamma_P(N)$, and since $\Gamma_P(N)$ is a triangulated graph, there are $b, c \in \Gamma_P(N)$ such that $ab, ac, bc \in P$. Thus, $D'(a) \subseteq V(b)$, and $\phi \notin D'(c) \subseteq V'(a) \cap V'(b) = \{P\}$, which gives $V'(b) = \text{Min}(N)$, a contradiction. Hence, $\text{Spec}(N)$ does not contain quasi-isolated points.

Conversely, suppose that $\text{Spec}(N)$ does not contain quasi-isolated points and take $a \in \Gamma_P(N)$. Then there are two different points $P, P' \in V(a) = D(P_a)$. Since $P_a \not\in P'$, there exists $z \in P_a$ such that $z \notin P'$. Also, there exists $y \in P$ with $y \notin P'$. Clearly, $zy \not\in P$ and $P \in V'(zy) = D'(P_a)$, which imply $P \notin \text{Supp}(zy)$. Thus $\text{Supp}(a) \cup \text{Supp}(zy) \neq \text{Spec}(N)$. Then by Lemma 3.2, there exists $c \in \Gamma_P(N)$ such that $\text{Supp}(a) \cup \text{Supp}(zy) \subseteq V(c)$, so by Theorem 3.3 (i), $c$ is adjacent to both $a$ and $zy$.

(ii) Let $\Gamma_P(N)$ be a hyper-triangulated graph. If $\text{Spec}(N)$ is not connected, then by Lemma 2.7, there exists an element $a \in \Gamma_P(N)$. Then $\text{Supp}(a) \cup \text{Supp}(-1 + a) = \text{Spec}(N)$, by Theorem 3.3, there is no vertex adjacent to both $a$ and $-1 + a$, a contradiction. The second part follows from Lemma 3.2 and Theorem 3.3.

Conversely, let $a \approx b$ be an edge in $\Gamma_P(N)$. Since $D(a) \cap D(b) = \phi$ and $\text{Spec}(N)$ is connected, $D(a) \cup D(b) \neq \text{Spec}(N)$. Thus by hypothesis, $\text{Supp}(a) \cup \text{Supp}(b) \neq \text{Spec}(N)$. Therefore, by Lemma 3.2 and Theorem 3.3, there exists a vertex adjacent to both $a$ and $b$.

(iii) Let $a \in \Gamma_P(N)$. Then there exists $b \in N \setminus P$ such that $ab \in P$. Since $2 \notin \Gamma_P(N)$, we have $2a \neq b$ and $a \neq 2b$. So $a, b, 2a$ and $2b$ are all distinct. Also, $ab, (2a)b, (2a)(2b)$ and $a(2b)$ belong to $P$. Hence $a, b, 2a$ and $2b$ is a cycle with length 4 containing $a$. □

As an immediate application of Theorem 3.7, we have the following corollary.

**Corollary 3.8** ([11], Theorem 3.1). Let $R$ be a commutative reduced ring. Then

(i) $\Gamma(R)$ is a triangulated graph if and only if $\text{Spec}(R)$ has no quasi-isolated points.

(ii) $\Gamma(R)$ is a hyper-triangulated graph if and only if $\text{Spec}(R)$ is connected space and for any $a, b \in \Gamma(R)$, we have that $ab \in P$ and $D(a) \cup D(b) \neq \text{Spec}(R)$ imply $\text{Spec}(R) \cup \text{Spec}(R) \neq \text{Spec}(R)$.

(iii) If $2 \notin \text{Z}(R)$, then every vertex of $\Gamma(R)$ is a 4-cycle vertex.

The next theorem will help to characterize all possible cycles in the ideal-based zero-divisor graph.
Theorem 3.9. Let $N$ be a 2-primal near-ring, $a, b \in \Gamma_{\mathcal{P}}(N)$ and $2 \notin \mathbb{P}$. If $2 \notin \Gamma_{\mathcal{P}}(N)$, then

(i) $c(a, b) = 3$ if and only if $D(a) \cap D(b) = \emptyset$ and $\text{Supp}(a) \cup \text{Supp}(b) \neq \text{Spec}(N)$.

(ii) $c(a, b) = 4$ if and only if either $D(a) \cap D(b) \neq \emptyset$ and $\text{Supp}(a) \cup \text{Supp}(b) \neq \text{Spec}(N)$, or $D(a) \cap D(b) = \emptyset$, and $\text{Supp}(a) \cup \text{Supp}(b) = \text{Spec}(N)$.

(iii) $c(a, b) = 6$ if and only if $D(a) \cap D(b) \neq \emptyset$ and $\text{Supp}(a) \cup \text{Supp}(b) = \text{Spec}(N)$.

Proof. (i) Follows from Lemma 3.2 and Theorem 3.3.

(ii) If $D(a) \cap D(b) = \emptyset$ and $\text{Supp}(a) \cup \text{Supp}(b) = \text{Spec}(N)$, there exists a path with vertices $a, b, 2a, 2b$, i.e., $c(a, b) \leq 4$. Now (i) implies that $c(a, b) = 4$. If $D(a) \cap D(b) \neq \emptyset$ and $\text{Supp}(a) \cup \text{Supp}(b) \neq \text{Spec}(N)$, then by Theorem 3.3, there exists $c \in \Gamma_{\mathcal{P}}(N)$ such that $c$ is adjacent to both $a$ and $b$. Thus, the path with vertices $a, c, b$ and $2c$ is a cycle with length 4.

(iii) If $c(a, b) = 6$, then parts (i) and (ii) imply that $D(a) \cap D(b) \neq \emptyset$ and $\text{Supp}(a) \cup \text{Supp}(b) = \text{Spec}(N)$. Conversely, let $D(a) \cap D(b) \neq \emptyset$ and $\text{Supp}(a) \cup \text{Supp}(b) = \text{Spec}(N)$. Then by Theorem 3.3, $d(a, b) = 3$. Also, (i) and (ii) implies that $c(a, b) \geq 4$. Hence, there are vertices $c$ and $d$ such that $ac, cd, bd \in \mathbb{P}$.

Now, if some vertex $e$ is adjacent to $b$, then $be \in \mathbb{P}$. Therefore, $\text{Spec}(N) = \text{Supp}(a) \cup \text{Supp}(b) \subseteq V(c) \cup V(e)$. However, $d(a, b) = 3$ implies that $a$ is not adjacent to $e$, i.e., $c(a, b) = 6$. If we consider the vertices $2c$ and $2d$, then we have a cycle with vertices $a, c, b, 2d$ and $2c$, i.e., $c(a, b) = 6$. □

From Theorem 3.9, we have the following corollary.

Corollary 3.10 ([11], Theorem 3.4). Let $R$ be a commutative reduced ring, $a, b \in \Gamma(R)$, and $2 \notin \Gamma(R)$. Then

(i) $c(a, b) = 3$ if and only if $D(a) \cap D(b) = \emptyset$ and $\text{Supp}(a) \cup \text{Supp}(b) \neq \text{Spec}(R)$.

(ii) $c(a, b) = 4$ if and only if either $D(a) \cap D(b) \neq \emptyset$ and $\text{Supp}(a) \cup \text{Supp}(b) \neq \text{Spec}(R)$ or $D(a) \cap D(b) = \emptyset$ and $\text{Supp}(a) \cup \text{Supp}(b) = \text{Spec}(R)$.

(iii) $c(a, b) = 6$ if and only if $D(a) \cap D(b) \neq \emptyset$ and $\text{Supp}(a) \cup \text{Supp}(b) = \text{Spec}(R)$.

As an immediate application of Theorem 3.9 or Corollary 3.10, we have the following corollary.

Corollary 3.11 ([11], Corollary 3.5). Let $R$ be a commutative reduced ring and $2 \notin \Gamma(R)$. Then every edge of a cycle with length 3 or 4.

Proof. Let $a \approx b$ be an edge in a cycle. Then $ab \in \mathbb{P}$ and $D(a) \cap D(b) = \emptyset$. If $\text{Supp}(a) \cup \text{Supp}(b) \neq \text{Spec}(R)$, then by Corollary 3.10, we have $c(a, b) = 3$. Otherwise, $\text{Supp}(a) \cup \text{Supp}(b) = \text{Spec}(R)$. Then by Corollary 3.10, we have $c(a, b) = 4$. □
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