GLOBAL ATTRACTORS FOR NONLINEAR WAVE EQUATIONS WITH NONLINEAR DISSIPATIVE TERMS

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Abstract. We show the existence, size and some absorbing properties of global attractors of the nonlinear wave equations with nonlinear dissipations like $\rho(x, u_t) = a(x)|u_t|^r u_t$.

1. Introduction

In this paper we are concerned with global attractors for the nonlinear wave equations with nonlinear dissipative term:

\begin{align}
(1.1) & \quad u_{tt} - \Delta u + \rho(x, u_t) + g(x, u) = f(x) \quad \text{in } \Omega \times R^+,
(1.2) & \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \quad \text{and } u(x, t)|_{\partial \Omega} = 0,
\end{align}

where $\rho(x, v)$ is a dissipation like $a(x)|v|^r v$ and $g(x, u)$ is a sourcing term like $|u|^\alpha u - |u|^\beta u, \alpha > \beta > 0$.

When $\rho(x, v) = v$, a linear dissipation, the existence of global attractor is well discussed and it is a standard result that if $0 < \alpha < 2/(N - 2)^+$, the problem admits a global attractor in the energy space $H^1_0(\Omega) \times L^2(\Omega)$. The proof is based on the exponential decay of energy of solutions for the case $g(x, u) = f(x) = 0$ and the compactness considered by Bavin and Vishik [2], Arrieta, Carvaho and Hale [1], Feireisl [6] etc. Finite dimensionality of global attractor is proved in Eden, Milani and Nickolenco [5]. See Ball [3] where the existence of global attractor is proved without uniqueness assumption and many references are cited.

Some standard results are generalized by several authors to the nonlinear dissipative cast $\rho(v)$ with $0 < \epsilon_0 \leq \rho'(v) < k_1 < \infty$. These are also based, at least in spirit, on the exponential decay for the equation with $g(x, u) = f(x) = 0$. For such nonlinearity finite dimensionality of the attractors is also investigated. See Raugel [16], Lasiecka and Ruzmaikina [11]. See also Chueshow, Eller and Lasiecka [4], where nonlinear boundary dissipation is considered.

However, when the dissipation has a stronger nonlinearity such as $\rho'(0) = 0$ or $\rho'(0) = \infty$ there seems to be few results. The object of this paper is to show the existence of global attractor in $H^1_0(\Omega) \times L^2(\Omega)$ for the essentially nonlinear case like $\rho(x, v) = a(x)|v|^r v, \quad r \neq 0$, and further give some characterizations of it. First we consider the case where the dissipation is effective in the whole domain $\Omega$ and

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next treat a more delicate case where the dissipation is effective possibly near some part of the boundary \( \partial \Omega \). We call the latter case as nonlinear localized dissipation. When the dissipative mechanism does not follow the Hooke’s law the most general dissipation would be of the form \( \rho(x, u) \) where \( \rho(x, v) \) is a monotone increasing function in \( v \). So, it seems an interesting problem and also reasonable to consider the nonlinear dissipative term like \( a(x)|u_t|^\alpha u_t \) as a typical model.

Lasiecka and Ruzmaikina [11] proved the existence of global attractor under the assumption that \( \rho(v) \) is strictly increasing and \( \rho'(v) \geq m_0 > 0 \) for \( |v| \geq 1 \), which is applied to the case \( \rho(v) = |v|^r, \quad 0 \leq r \leq 4/(N-2)^+ \). But, our result includes the case \(-1 < r < 0\) and gives precise informations on the size and the absorbing rate. Feireisl and Zuazua [7] treated the equation with nonlinear localized dissipation \( \rho(x, v) = a(x)\rho_0(v) \) and the terms \( g(x, u) = g(u), \quad f(x) = 0 \) and proved the existence of global attractor under the assumption that \( \rho_0(v) \) is strictly increasing in \( v \) and \( 0 < m_0 \leq \rho'(v) \leq m_1 < \infty \) for \( |v| > 1 \). But, no estimate on the size and the absorbing rate is given there.

2. Notations and main results

Let us state precise assumptions on the terms \( \rho(x, u), \quad g(x, u) \) and \( f(x) \). We first assume the following.

**Hyp.A.** \( \rho(x, v) \) is measurable in \( x \in \Omega \) for any \( v \in R \) and differentiable in \( v \neq 0 \) for a.e. \( x \in \Omega \), and satisfies

\[
\begin{align*}
(2.3) & \quad k_0|v|^{r+2} \leq \rho(x, v)v \leq k_1|v|^{r+2} \quad \text{if} \quad |v| \leq 1, \\
(2.4) & \quad k_0|v_1 - v_2|^{r+2} \leq (\rho(x, v_1) - \rho(x, v_2))(v_1 - v_2) \\
& \quad \leq k_1(|v_1 - v_2|^2 + |v_1 - v_2|^{r+2}) \quad \text{if} \quad |v_1|, |v_2| \leq 1
\end{align*}
\]

and

\[
\begin{align*}
(2.5) & \quad k_0|v_1 - v_2|^{p+2} \leq (\rho(x, v_1) - \rho(x, v_2))(v_1 - v_2) \\
& \quad \leq k_1(|v_1|^p + |v_2|^2)|v_1 - v_2|^2 \quad \text{if max} \{ |v_1|, |v_2| \} \geq 1,
\end{align*}
\]

where \( k_0, k_1 > 0, \quad -1 < r < \infty \) and \( 0 \leq p \leq \frac{4}{(N-2)^+} \).

**Hyp.B.** \( g(x, u) \) is measurable in \( x \in \Omega \) for all \( v \in R \) and continuous in \( v \in R \) for a.e. \( x \in \Omega \), satisfying:

\[
(2.6) \quad g(x, 0) = 0, \quad g(x, u)u + \bar{L} \geq \mu(\int_0^u g(x, \eta)d\eta + L) \geq 0
\]

for some \( \mu > 0 \) and \( L, \quad \bar{L} \geq 0 \) and

\[
(2.7) \quad |g(x, u_1) - g(x, u_2)| \leq k_1(1 + |u_1|^{\alpha} + |u_2|^{\alpha})|u_1 - u_2| \quad \text{for} \quad u_1, u_2 \in R,
\]

with some \( k_0, k_1 > 0 \) and \( 0 \leq \alpha < 2/(N - 2)^+ \).

We set

\[
G(x, u) = \int_0^u g(x, \eta)d\eta.
\]

**Hyp.C.** \( f \in L^2(\Omega) \).
It is well known that under Hyp.A, Hyp.B and Hyp.C the problem (1.1)-(1.2) admits a unique solution \( u \in C([0, \infty); H^1_0(\Omega)) \cap C([0, \infty); L^2(\Omega)) \) for each \((u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)\). We denote the solution \( u(t) \) by \( U(t)(u_0, u_1) \). Since our system is autonomous \( U(t) \) is a continuous group as operator in \( H^1_0 \times L^2 \). Our result is real as follows.

**Theorem 1.1.** Under Hyp.A, Hyp.B and Hyp.C the problem has a global attractor in the space \( H^1_0(\Omega) \times L^2(\Omega) \). Further \( A \) is include in a ball \( B(\bar{R}) \) in \( H^1_0(\Omega) \times L^2(\Omega) \) centered at 0 with the radius \( \bar{R} = C(M_0^2 + L + \bar{L}) \) and for any bounded set \( B_0 \subset H^1_0(\Omega) \times L^2(\Omega) \) we have the absorbing property

\[
\text{dist}(U(t)B_0, B(\bar{R})) \leq C(B_0)(1 + t)^{-1/2}\gamma
\]

where \( \gamma \) is defined by

\[
\gamma = -r/2(r + 1) \text{ if } -1 < r \leq 0,
\]

\[
r/2 \text{ if } r \geq 0.
\]

We note that when \( f = 0 \) and \( L = \bar{L} = 0 \), the ball \( B(\bar{R}) \) is reduced to 0 and the estimate gives a well-known decay rate of solutions. For the proof of Theorem 1.1 we use an idea in our earlier paper [13] where the same algebraic decay or the estimate gives a well-known decay rate of solutions. For the proof of Theorem 1.1 we use an idea in our earlier paper [13] where the same algebraic decay or stability of the bounded solution in proved for the case \( g(x, u) = 0 \). In Lasiecka and Ruzmaikina [11] the existence of global attractor is proved for a similar problem, but, the size of the absorbing set nor the absorbing rate as (1.8) is not derived there.

Secondly, we consider a more delicate case where \( \rho(x, v) \) is nonlinear and possibly vanishes on some large area in \( \Omega \). To state our assumption on the dissipation \( \rho(x, v) \) precisely, we define a set of the boundary \( \Gamma(x_0) \) introduced by Russell [17]:

\[
\Gamma(x_0) = \{ x \in \partial\Omega | (x-x_0) \cdot \nu(x) > 0 \},
\]

where \( x_0 \in \mathbb{R}^N \) and \( \nu(x) \) is the outward normal vector at \( x \in \partial\Omega \). Let \( x(x) \) be a nonnegative bounded function on \( \Omega \) satisfying:

There exist \( x_0 \in \mathbb{R}^N \) and a relatively open set \( w \subset \bar{\Omega} \) such that

\[
\Gamma(x_0) \subset w \text{ and } a(x) \geq c_0 > 0 \text{ for } x \in w.
\]

**Theorem 1.2.** Under Hyp.A, Hyp.B and Hyp.C, the problem has a global attractor \( A \) in the space \( H^1_0(\Omega) \times L^2(\Omega) \). Further, under Hyp.B, (1), \( A \) is included in a ball \( B(\bar{r}) \) centered at 0 with a radius \( \bar{R} = \bar{R}(M_0, K_0, d_0, d_1) > 0 \) and it holds that

\[
\text{dist}(U(t)B_0, B(\bar{R})) \leq C(B_0)e^{-\lambda t},
\]

where \( \lambda \) depends on \( B_0 \) except for special cases.

Under Hyp.B, (2), \( A \) is included in a ball \( B(\bar{R}) \) centered at 0 with a radius \( \bar{R} \) given

\[
\bar{R}^2 = C(M_0^2 + K_0^2), C > 0
\]
and it holds that

\[(2.9) \quad \text{dist}(U(t)B_0, B(\tilde{R})) \leq C(B_0)(1 + t)^{-1/2}\gamma\]

where \(\gamma\) is the same as in Theorem 1.1.

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