Some open problems in \( q \)-series

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1 Introduction

A finite nonincreasing sequence of positive integers, \( \lambda = \lambda_1 \lambda_2 \cdots \lambda_l \), is called an integer partition of \( n \), where \( n = \sum_{i=1}^{l} \lambda_i \). We follow standard notations in [1]. The length of a partition \( \lambda \) is the number of its parts, denoted \( l(\lambda) \). Let \( |\lambda| \), called the size of \( \lambda \), denote the number which \( \lambda \) is a partition of. Let \( p(n) \) denote the number of all partitions of \( n \). Then we have

\[
\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_\infty},
\]

where \( (a; q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i) \). Let \( (a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i) \).

In the following sections we present some open problems proposed by George Andrews while he has been visiting KAIST from August 10 to August 23, 1997.

2 Alder conjecture

Define two sequences \( \{C_d(n)\}_{n \geq 0} \) and \( \{c_d(n)\}_{n \geq 0} \) as follows:

\[
\sum_{n=0}^{\infty} \frac{q^d(\binom{d}{2})+n}{(q; q)_n} = \sum_{n=0}^{\infty} C_d(n)q^n,
\]

\[
\frac{1}{(q; qq^{d+3})_{\infty}(q^{d+2}; q^{d+3})_{\infty}} = \sum_{n=0}^{\infty} c_d(n)q^n.
\]
Conjecture 2.1 \[ C_d(n) \geq c_d(n) \text{ for each } d \geq 1 \text{ and each } n \geq 0. \]

It is not difficult to show that \( C_d(n) \) is the number of partitions of \( n \) with differences between parts \( \geq d \) and \( c_d(n) \) is the number of partitions of \( n \) with parts \( \equiv 1 \text{ or } d + 2 \text{ mod } d + 3 \). The case \( d = 1, C_1(n) = c_1(n) \), is the Euler identity, i.e.

\[
(-q; q)_\infty = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q; q^2)_{\infty}},
\]

and the case \( d = 2, C_2(n) = c_2(n) \), is the first Rogers-Ramanujan identity, i.e.

\[
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q; q^5)_{\infty}(q^4; q^5)_{\infty}},
\]

and the case \( d = 3, C_3(n) \geq c_3(n) \), is the Schur identity.

The conjecture is proved for \( d = 2^s - 1, s \geq 4 \) and the inequality is proved for any fixed \( d \), if \( n \) is sufficiently large.


3 Borwein conjecture

Peter Borwein proved that for any prime \( p \), the coefficients \( a_n \) and \( a_{n+p} \) have the same sign, where

\[
\frac{(q; q)_\infty}{(q^p; q^p)_\infty} = \sum_{n=0}^{\infty} a_n q^n.
\]

Borwein observed the following:

Conjecture 3.1 If

\[
\frac{(q; q)_3n}{(q^3; q^3)_n} = A_n(q^3) - qB_n(q^3) - q^2C_n(q^3),
\]

then each polynomial \( A_n(x) \), \( B_n(x) \) and \( C_n(x) \) has nonnegative coefficients.

We can extend this to the following:

Conjecture 3.2 If

\[
(1 - q)(1 - q^2)\frac{(zq^3; q^2)_3n}{(zq^3; q^3)_n} = A_n(z, q^3) - qB_n(z, q^3) - q^2C_n(z, q^3),
\]

then each polynomial, in \( x \) and \( z \), \( A_n(z, x) \), \( B_n(z, x) \) and \( C_n(z, x) \) has nonnegative coefficients.

In addition to this, the same thing happens with the prime 5.
Conjecture 3.3 Let
\[
\frac{(q; q)_n}{(q^5; q^5)_n} = A_n(q^5) - q B_n(q^5) - q^2 C_n(q^5) - q^3 D_n(q^5) - q^4 E_n(q^5).
\]

Then each polynomial \( A_n(x) \), \( B_n(x) \), \( C_n(x) \), \( D_n(x) \) and \( E_n(x) \) has nonnegative coefficients.

The primes 3 and 5 are special. 7 doesn’t give a nice generalization.

References:
2. D. Bressoud and D. Stanton

4 Rogers-Ramanujan identities

Rogers-Ramanujan identities assert that
\[
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q, q^2; q^5)_\infty},
\]
\[
\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = \frac{1}{(q^2, q^3; q^5)_\infty}.
\]

The first identity implies that the number of \( n \) into parts that differ by at least 2 equals the number of partitions of \( n \) into parts that are 1 or 4 mod 5. There is a combinatorial proof of this by Garcia and Milne, using the involution principle.

Problem. Is there a bijective proof of this identity that doesn’t rely on the involution principle?

5 More open problems

1. Problem. Can you find infinitely many pairs \((S, T)\) of sets \( S \) and \( T \) of positive integers so that for each \( n \geq 1 \), the number of partitions of \( n \) into elements in \( S \) equals the number of partitions of \( n - 1 \) into elements in \( T \). In other words
\[
\prod_{i \in S} \frac{1}{1 - q^i} = 1 + \prod_{i \in T} \frac{q}{1 - q^i}.
\]

2. In the paper, Invent. Math. 91 (1988), 391–407, it is shown that
\[
\sum_{n=0}^{\infty} \frac{q^{\frac{n+1}{2}}}{(-q; q)_n}
\]
has asymptotically all coefficients a zero. However every integer appears as a coefficient infinitely many times.

**Problem.** *Is there a simple combinatorial proof of this?*

**References**

