THE INFLUENCE OF TECHNOLOGY ON MATHEMATICS INSTRUCTION: CONCERNS AND CHALLENGES

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Abstract. Given the increasing uses in education of rapidly evolving technologies, some of which include animation and many of which have embedded computer algebra systems, we examine how technology can or will influence the ways mathematics is taught, learned, used, and created. This focus on technology provides an important context for addressing critical issues that lie at the heart of mathematics instruction.

“Computer Algebra Systems: Issues and Inquiries” [1] illustrates how computer algebra systems (CASs) can enable students to comprehend rich structures, visualize complicated relationships, breathe life into static representations, pursue limiting cases, and make intuitive leaps. It also includes examples that show a variety of ways that CASs can constrain, misdirect, or in other ways influence how students think and learn mathematics. A major theme of the article was Page’s concern that CASs can vitiate conceptual understanding if they are permitted to foster an immediacy towards, or overdependence on, computation. CASs can also preclude or thwart creative thinking if we allow them to anesthetize our impulses to consider other representations, to seek new relationships lurking in representations, and to be innovative in how we process information.”

In a response to Page’s paper, David Smith writes [2],

“. . . The possible evils he sees in the use (or abuse) of CAS technology are already evils in our educational system, even without technology. . . . Whether we are discussing blackboards or computers, it is not the tools that create these distortions of education. The real threat posed by the availability of more powerful tools is that they will enable to educational establishment to scale new heights in its lemming-like drive to replace education with training . . .”

Given the increasing uses in education of rapidly evolving technologies, some of which use animation and many of which have embedded CAS capabilities, we here re-examine some earlier issues and consider further how technology can or will influence the ways mathematics is taught, learned, used, and created at all levels. This focus on technology provides a context for addressing critical issues that lie at the heart of mathematics instruction.

1. FUNDAMENTAL ASSUMPTIONS.

Everything in this paper is predicated on the belief that the two most critical factors in teaching mathematics concern “what” one conveys and “how” we do so. Both factors are intimately intertwined with information-processing and learning; each has affective as well as cognitive dimensions.

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What we communicate in mathematics transcends the elucidation of mathematical concepts: we also convey (consciously and unconsciously) a great deal to students about the intrinsic nature and value of the discipline itself.

How we teach mathematics also goes beyond the exchange of ideas and information. Our actions and instructional demands on students imprint them with long lasting psycho-social values on what it means to do mathematics and who should do it.

The remarks and simple examples in this paper illustrate and exemplify what is succinctly articulated by Davis and Anderson [3].

“Mathematics has elements that are spatial, kinesthetic, elements that are arithmetic or algebraic, elements that are verbal, programmatic. It has elements that are logical, didactic, and elements that are intuitive or even counterintuitive. . . . These may be compared to different modes of consciousness. To place undue emphasis on one element or groups of elements upsets a balance. It results in an impoverishment of the science and represents an unfulfilled potential.”

Today, it is well known that there exists major differentiation of functions between the brain’s left and right hemispheres. In the most simplistic terms, the left-hemispheric thinking resembles the discrete, sequential processing of a digital calculator whereas right-hemispheric thinking simulates the concurrent, relational activity of an analog computer. Compare, for example, the proof by induction with Gauss’s relational proof (Figure 1) that \[ 1 + 2 + \cdots + n = \frac{n(n+1)}{2}. \] (Would Gauss have thought of “pairings” if he had been weaned on a numeric calculator?) If the dots on the right triangle’s hypotenuse in Figure 2 are labeled \( a \) and all the other dots are labeled \( d \), we obtain the (right-hemispheric) representation [4] that the sum of the first \( n \) terms of an arithmetic progression plus \( n^2 \) times the difference equals the sum of the next \( n \) terms of the progression.

Students with diverse career objectives (science, engineering, statistics, computers, liberal arts, etc.) have differing predispositions toward learning and using mathematics. Thus, because there are different modes of cognitive functioning, mathematics instruction is more effective when several modes of thinking are used.

We are much more explicit in enunciating principles than in when and how they are applied. Formulas and theorems, for example, do not always carry internal information about contexts or situations that should evoke their use. Students...
usually adopt the representation of a problem from the language of the statement rather than search for a more efficient representation of the problem. For example, consider the following.

How many students would relate “hypotenuse $c$” to the rectangle’s other diagonal (the circle’s radius)? Although one could get the answers in Figure 4 by a (left-hemispheric) solution of the simultaneous equations, moving the unit circle’s center along the x-axis would be a quicker (right-hemispheric) way to see where the circle intersects the lines $y = \pm x$. [It will be interesting to see if the increased instructional uses of computer animations will increase student’s inclinations to use more dynamic representations and ways of thinking.] Figure 5a depicts a student’s solution based on the assumption that the result for the inscribed star was true for every 5-pointed star. The author’s own solution was obtained by rotating a paper arrow (vector).

It should be clear that a higher order-awareness is needed to be able to recognize and apply concepts in different relational contexts. Thus, instructors need to help and encourage students to develop their skills in reformulating and restructuring problem representations.

For another solution, the author imagined the triangle’s sides were rigid extensible rods pinned at the vertices or at the “elbows” (points where the sides intersect). By a sequence of vertex movements, none of which changed the star’s vertex sum, it was possible to transform one five-pointed star to another with the same vertex sum. The interesting combinatorics-topological results discovered are described in [5]. Students may also learn and discover interesting things by playful hands-on
exploration. Although technology today can quickly display such results (say, envelopes of curves), students may enjoy and prefer the related hands-on activities (paper-folding methods) for producing them. As Page writes in [1]:

“Neither CASs, nor any other prescribed representational medium, can be (or should be attempted to be) the sole means to accommodate the rich and diverse ways we process information, formulate conjectures, and attempt to solve problems . . . . This footnote argues for the creation of mathematics laboratories that make use of a wide variety of physical objects and computational processes through which students can discover mathematical principles and build intuitions upon which CASs can further enhance and develop in meaningful ways.

2. Misperceptions and misconceptions.

As Figures 3 and 4 illustrate, students do not always see or interpret what is obvious to, or intended by, instructors. This should not be surprising given that what we perceive depends on our experiences and cognitive structures. Even the simplest of computer-displayed graphs can be misperceived by students. For instance, as Schoenfeld [6] shows, the vertically translated graphs in Figure 6 appear to get closer as \( x \) increases. After explaining and dispelling this illusion (by measuring vertical segments joining the two curves), he concludes, “We can’t assume that students will see what we want them to see, even if it’s accurately represented on the screen.” Orton [7] asked 110 calculus students what happens to the secants \( PQ \) on a sketched curve (Figure 7) as the point \( Q_n \) tends toward \( P \). Forty-three students could not answer this even after being prompted to see the result. Typical responses were “The line gets shorter” and “It becomes a point”. Students apparently focused on the wording “secants \( PQ \)”. Would the responses have been better responses if the secant lines are extended, or if a dynamic representation were presented?

In Visual Thinking [8], Arnheim questions what learners see in a textbook diagram or film.

“Have we the right to take for granted that a picture shows what it represents, regardless of what it is like and who is looking?” . . . “Careful investigation of what the persons for whom these images are made see in them is indispensable.”
Good advice! To improve and enhance our instructional visualizations and created animations, we should nominate our students to be beta testers who describe what they perceive and understand.

Computing carries with it the potential for various types of misperceptions. These can be sources of confusion for students or opportunities for instructors to enrich mathematical thinking.

**Example 1.** (a) Because of roundoff error, the graphical and numerical depictions of $\sum \frac{1}{k}$ as $n$ increases will corroborate that the harmonic series converges. After explaining this misperception, here is the chance for an instructor to illustrate another way of thinking – as, for example, the following analogical proof for the harmonic series diverges.

If $S = \sum_1^\infty \frac{1}{k}$ for some real number $S$, then $S = \sum_1^\infty \left( \frac{1}{2k-1} + \frac{1}{2k} \right) > \sum_1^\infty \frac{1}{k} = S$ yields the contradiction $S > S$.

(Would Euclid equipped with a CAS have thought of his analogical proof of the infinitude of the primes?)

(b) Students who use implicit differentiation to compute $y' = \frac{-x}{y}$ for $x^2 + 1 + y^2 = 0$ can easily confirm that their answer is the same as $y' = -x\sqrt{-1 - x^2}$ displayed by a CAS. Since the equation $x^2 + 1 + y^2 = 0$ does not define a real-values function $y = f(x)$, the derivative does not exist. (Although the CAS is interpreting $y = -x\sqrt{-1 - x^2}$ and its derivative as complex-valued functions, this is not accessible to students who have no knowledge of complex function theory.) Here, however, is an example instructors can use to emphasize to students their need to establish before calculating a function’s derivative that the function is differentiable.

(c) A more serious misperception of doing mathematics is students’ uses of a computer to solve a problem by examining all possible test cases – as, for example, the response to “Determine the highest power of 5 that divides 50!, where $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$.”

Solution: “Enter the expression $50!/5^m$, then substitute several guesses for $m$ using the $\cdots$ command. You will know that you have the right answer when your quotient is an integer and yet any larger power yields a fractional part.” (It would be interesting to speculate what that student would have written based on the hint “What power of 5 divides each number from 1 to 50?”)

The following example illustrates the extent to which computing enables students to apply that which is not well understood.

**Example 2.** For a data set $\{(x_i, y_i)\}$ that has large $x$-values, students are often told to use the $\{(X_i, y_i) = (x_i - \alpha, y_i)\}$ data to model and make predictions about the $\{(x_i, y_i)\}$ data. If regression produces the function $y(x)$ that models the $\{(x_i, y_i)\}$data and the same type regression yields the function $Y(X)$ for the $\{(X_i, y_i)\}$ data, does $Y(x_i - \alpha) = y(x_i)$ for each $x_i$? The answer, of course, depends on the type of regression used. However, many student activities and explorations that involve regression do not mention this, nor do they (or their teachers) explain to students why they may observe ERROR, OVERFLOW, or other unexpected messages.

Since the scatter plot of the $\{(X_i, y_i) = (x_i - \alpha, y_i)\}$ data is the scatter plot of the $\{(x_i, y_i)\}$ data horizontally translated $\alpha$ units, both scatter plots have the same spatial configuration. Therefore (Figure 8), for linear and polynomial regression,
\[ Y(x_i - \alpha) = y(x_i) \] for each \( x_i \). This is also true for exponential regression via linear regression of the \((x_i, \ln y_i)\) data, which can be seen by relabeling \( Y(X) \) and \( y(x) \) as \( \ln Y(x) \) and \( \ln y(x) \). Since \( Y(X_i) = \ln y(x_i) \), we have \( Y(X_i) = y(x_i) = e^{B(e^A)^x} \) for each \( x_i \). For data involving large \( x \)-values, \((e^A)^x\) is very large (and may cause overflow) when \( A > 0 \), and \((e^A)^x\) is very small (and roundoff to zero may produce an error message) when \( e^A < 1 \).

Figure 9 shows why power regression on shifted \(\{(x_i - \alpha, y_i)\}\) data depends on the choice of \( \alpha \). Since the logarithm compresses large values much more than it does small values, the \((\ln x_i, \ln y_i)\) scatter plot will be horizontally compressed more than the \((\ln (x_i - \alpha), \ln y_i)\) scatter plot. And, since both scatter plots have the same \( \ln y_i \) values, the slope \( b \) of the best linear fit for the \((\ln x_i, \ln y_i)\) data is greater than the slope \( b_1 \) of the best linear fit for the \((\ln (x_i - \alpha), \ln y_i)\) data. Therefore, the power function \( y(x) \) that models the \(\{(x_i, y_i)\}\) data has a larger exponent than the power function \( Y(X) \) that models the \((\ln (x_i - \alpha), \ln y_i)\) data, and so \( Y(x_i - \alpha) \neq y(x_i) \) for every \( x_i \). For very large \( x \)-values, the best fit line \( y = b(\ln x) + c \) may look almost vertical – in which case \( b \ln x \) is very large and \( c \) is a very small negative number. Because of this, a utility may display an overflow message for \( x^b \), and display 0 or an error message due to \( e^c \).

The explanations above should be accessible to students who are asked to perform regression. Following this discourse in [9], students are asked to show, or explain why, logarithmic regression is not translation invariant, and in a related exploration students are guided to discover what can be said about regression on scaled \(\{(x_i/\beta, y_i)\}\) data for \( \beta > 1 \). As this example demonstrates, we must help students understand what they are computing and why before asking them to do so.

In [10] Paul Zorn laments:

“Compared with exact or closed-form methods, approximate, numerical, iterative, and recursive techniques get too little attention. ... Because closed-form methods fail with so many innocuous-looking problems, exercises and applications are carefully contrived, and they show it. Arc-length integral problems are especially ludicrous; they can almost never be computed in a closed form. An excellent opportunity to use numerical integration where it is needed is wasted.”

Of course Zorn is correct. But wait! As in Example 2, we must be sure that students understand what they are computing before they are asked to do so.
Example 3. Although students may use numerical integration to compute \( L = \int_a^b \sqrt{1 + (f'(x))^2} \, dx \), there is no reason to claim that \( L \) is the actual arc length of \( y = f(x) \) from \( x = a \) to \( x = b \). Briefly, for every natural number \( n \), the length \( L_n \) of the inscribed piecewise linear path with \( n \) segments is bounded above by the arc length \( L \) to be computed. Therefore, \( L \) is an upper bound for the sequence \( \{L_n\} \) of approximating paths’ lengths. Since \( \{L_n\} \) is monotonically non-decreasing as \( n \) increases, the value \( L \) is the least upper bound for \( \{L_n\} \). But how do we know that this least upper bound \( L \) really is the arc length \( L \)? The increasing physical proximity to a curve of approximating paths (as \( n \) increases and the partition’s mesh tends to zero) does not necessarily imply better numerical approximations to the curve’s true length. (For instance, the curve \( y = x \) from \( x = 0 \) to \( x = 1 \) has arc length \( \sqrt{2} \), but every inscribed polygonal path consisting of horizontal and vertical segments has length 2 no matter how close such an inscribed polygonal path is to this linear segment.) This logical gap in the development of the formula for arc length appears to have eluded a sizable proportion of the mathematics community. This same incompleteness exists in other applications of the definite integral. For instance, there is no justification in most texts that the definite integrals for surface area in three dimensions do indeed actually measure the actual surface area. In [11], Page shows how these logical gaps can be eliminated.

3. INFLUENCES OF TECHNOLOGY ON REASONING AND PROBLEM SOLVING.

Mathematical reasoning predicated on knowledge of a solution’s existence may be quite different from reasoning that does not presuppose existence. Compare, for example, Leibniz’s proof (Figure 10) that the geometric series converges for \( 0 < r < 1 \) with the proof (Figure 11) based on the fact that the sum of the series is a real number, \( S \).

The following example illustrates how computer processing can influence students’ reasoning and attempts to solve problems.

Example 4. For \( x > 0 \), what is the number \( \sqrt{\ldots \sqrt{x}} \)? In the rush to compute, students using test cases etc., may too readily conclude that the repeated square roots of a positive number eventually yields the number 1. No thinking is required, and no insights result.

Using the fact that \( \sqrt{\ldots \sqrt{x}} \) is a real number – say, \( y \), one might write \( y^2 = y \) and obtain \( y = 1 \). Thus, observing the existence of a solution (limit) enables one to
determine its value. Instructors can provide more insight to the problem by showing, or leading students to observe, that

$$\sqrt{x} = x^{1/2}, \quad \sqrt[3]{x} = x^{1/3}, \ldots, \sqrt[n]{x} = x^{1/n} \rightarrow x^0 = 1$$

as \(n\) increases. Here also is an opportunity to display a cobweb diagram that shows how the iterates of \(f(x) = \sqrt{x}\) converge to the fixed point of \(f, x = 1\). Curious students who display the successive graphs of

\[
f_1(x) = \sqrt{x}, \quad f_2(x) = \sqrt{x + f_1(x)}, \quad f_3(x) = \sqrt{x + f_2(x)}, \text{ and so on,}
\]

will observe that the graphs converge to the graph of some function \(F(x)\) for \(x > 0\). As above, they can use (or be led to use) this observation to obtain \(F(x) = (1 + \sqrt{1+4x})/2\). What additional insight can instructors impart? How might instructors respond to students who plot the data \(\{(x, f_9(x)) : x = 1, 2, \ldots, 8\}\) and use quadratic regression to claim that “\(F(x)\) is closely approximated by the function \(y = -0.01517x^2 + 0.38132x + 1.27562\)?

In [10], Hosack argues that “numerical integration can be developed as the ‘norm’ or ‘standard’ with closed form integration being considered as a special case.” Although this may have merit for those oriented toward analysis, one must naturally wonder how this emphasis might affect students’ propensities to discover patterns, interrelate concepts, and engage in creative explorations.

**Example 5.** Given the new norm of numerical integration, how many students who compute

$$\int_{0}^{\pi/2} \sin^2 x dx \approx 0.785398164$$

would be motivated to compute \(\int_{0}^{\pi/2} \sin^4 x dx \approx 0.589048623, \int_{0}^{\pi/2} \sin^6 x dx \approx 0.490873852,\) or any other such integral? How many would students be likely to look for a closed-form representation of 0.785398164? It need not be “something gained, something lost.” For instance, asking students to express 0.785398164 in terms of \(\pi/2\) would lead them to discover (on dividing 0.785398164 by \(\pi/4\)) that

$$\int_{0}^{\pi/2} \sin^2 x dx \approx \pi/4$$

This, or the instructor’s queries, can motivate them likewise to discover that

$$\int_{0}^{\pi/2} \sin^4 x dx \approx 3\pi/16 \quad \text{and} \quad \int_{0}^{\pi/2} \sin^6 x dx \approx 5\pi/32$$

These closed-form integrals immediately suggest the readily proved recursive relation

$$\int_{0}^{\pi/2} \sin^{2n} x dx \approx \frac{2n-1}{2n} \int_{0}^{\pi/2} \sin^{2n-2} x dx$$
Letting \( A_{2n} = \int_0^{\pi/2} \sin^{2n} x \, dx \), so that \( A_{2n} = \frac{2n-1}{2n} A_{2n-2} \), students would obtain
\[
A_2 = \frac{\pi}{2} \cdot \frac{1}{2}, \quad A_4 = \frac{\pi}{2} \cdot \frac{1.3}{4}, \quad A_6 = \frac{\pi}{2} \cdot \frac{1.3.5}{2.4.6},
\]
and be led to Wallis’ sine formula
\[
\int_0^{\pi/2} \sin^{2n} x \, dx = \frac{\pi}{2} \cdot \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}.
\]

Here is another illustration of what might, or might not, be discovered using technology.

**Example 6.** “Of all pairs of positive numbers having a given product, \( P \), which pair has the smallest sum?

Students using a numerical or graphing calculator can observe the answer for specific values \( P \), and thereby conclude (Figure 12) that the smallest sum is \( S_{\text{min}} = 2\sqrt{P} \) when \( x = y = \sqrt{P} \). Students who calculate (or use a CAS to display) the derivative of \( s = x + P/x \) can readily obtain \( S_{\text{min}} \). There is not much thinking and no insight in these solutions. [At the very least, precalculus students could be asked to prove (or be shown how to prove) their answer by using the arithmetic-geometric mean inequality, and calculus students should be asked to explain before doing anything how they know that \( S \) actually has a minimum value.

Without motivation or guidance, it is unlikely that students with “compute the answer” mentality will shift to an “explore the answer” mode of thinking. Here is an opportunity for instructors to use dynamic computer representations of the problem — to show, for example (Figure 13), that the minimum sum must occur when and only when the moving line \( S = x + y \) is tangent to the fixed hyperbola \( xy = P \). Reversing directions also shows that the minimum product of two positive numbers having a given sum \( S \) must occur when and only when the moving hyperbola \( xy = P \) is tangent to the fixed line \( S = x + y \). (See also [11].) Figure 13 suggests that the result carries over to three positive numbers, and this can be demonstrated by showing that the minimum sum occurs when and only when the movable plane \( S = x + y + z \) is tangent to the fixed surface \( xyz = P \). Now curious calculus students might want to (or be asked to) investigate how the problem can be further extended.

The lure of technology can lead to demonstrations or proofs that are devoid of insight. This is especially unfortunate if such results can be proved in ways that enrich students’ mathematical understandings. The following, taken from a CAS
newsletter, describes how an instructor used a computer algebra system to prove the reflective property of a parabola in a high school class whose students knowledge of trigonometry was limited.

In Figure 14, the light ray is reflected by the parabola \( y^2 = 4x \). Sometime or other we have heard that all reflected rays meet in the focus. Now the curve’s tangent at \( P(x, 2\sqrt{x}) \) takes the role of a mirror, with \( y' = 1/\sqrt{x} \). Use this to show that the rays reflected at points \( A(2, y) \) and \( B(6, y) \) meet at the same point on the \( x \)-axis.

General solution: \( \alpha = \tan^{-1}\left(\frac{1}{\sqrt{x}}\right) \) and \( \alpha' = 2\alpha = 2\tan^{-1}\left(\frac{1}{\sqrt{x}}\right) \), which gives the reflected line’s slope \( m = \tan(2\tan^{-1}\left(\frac{1}{\sqrt{x}}\right)) \). Even without any sum rule for the trigonometric functions, CAS is helpful.

(CAS output) \( \tan\left(2 \cdot \tan^{-1}\left(\frac{1}{\sqrt{x}}\right)\right) = \frac{2\sqrt{x}}{x-1} \)

Thus, the family of lines passing through the points \( P\left(x_0, 2\sqrt{x_0}\right) \) is \( y - 2\sqrt{x_0} = \frac{2\sqrt{x_0}}{x_0-1} (x - x_0) \). Setting \( y = 0 \) and solving for \( x \), we find that \( x = 1 \). All reflected rays meet in the focus \( F(1,0) \). Figure 15 shows the instructor’s included CAS display.

Does Figure 15 provide any more insight than the proof why this result is true? How many students could explain what exactly causes this result to be true for this parabola, and how many could say why this property is true for all parabolas? The intent of these remarks is not to disparage an instructor’s well-intentioned efforts, but rather to show how much more could have been gained by introducing and building on fundamental concepts.

**Example 7.** Computer-generated displays (as in Figure 15) can motivate students to ask if rays parallel to the \( y \)-axis reflect off the parabola \( y = ax^2 \) and pass through its focus. Since the students knew (or could be shown visually) that the parabola becomes indistinguishable from the tangent line through a point \( P\left(x_0, ax_0^2\right) \) as \( x \) approaches \( x_0 \), here is an opportunity to show students how to calculate the equation of the tangent at \( x = x_0 \). Writing \( y = ax^2 \) as

\[
y = a \left\{ x_0 + (x - x_0) \right\}^2 = ax_0^2 + 2ax_0(x-x_0) + a(x-x_0)^2
\]

and ignoring the term \( a(x-x_0)^2 \), which is very small when \( x \) is close to \( x_0 \), we find that the tangent to the graph \( y = ax^2 \) at \( P(x_0, y_0) \) has the equation

\[
y = 2ax_0(x-x_0) + y_0
\]

Note, in particular, the tangent’s \( y \)-intercept is \( Q(0, -y_0) \).
Now suppose we prove or visually demonstrate that the parabola with equation \( x^2 = 4py \) has focus \( F(0, p) \). Then, using the tangent’s equation, it becomes clear that the relation between the parabola’s equation and coordinates of its focus makes \( |FP| = |FQ| \):

\[
|FP| = \sqrt{(y_0 - p)^2 + x_0^2} = \sqrt{y_0^2 - 2py_0 + p^2 + 4py_0} = \sqrt{(y_0 + p)^2} = |FQ|
\]

Thus, \( \beta = \alpha \) and ray \( PR \) is parallel to the \( y \)-axis.

Students likewise can show why this property is true for the parabola \( y^2 = 4px \). Using translations of the graphs \( y = \frac{1}{4}x^2 \) and \( x = \frac{1}{4}y^2 \), students can prove, or be led to prove, that the reflection property is true for every parabola \( y = ax^2 + bx + c \) and \( x = ay^2 + by + c \). Based on such understandings, students can use CASs to explore their conjectures, as well as confirm the reflection property for every parabola.

The informed, judicious use of technology can dramatically enhance the ways mathematics is taught, learned, used, and created at all levels, from primary school, to college/university. However, to realize its fullest potential we need to address a wide variety of issues, some of which are briefly considered in the next section. See [1] for a much extended exposition of these and related issues.

4. **Impact on students, faculty, and the parameters of instruction.**

It is well known that a large constellation of behavioral and social factors are at work in predisposing students to success or failure in mathematics courses.

**Instructional environments.** The quality and quantity of interaction in instruction are important ingredients for learning. Some students prefer to work alone, some do better in small groups, and others learn best through different dynamics and forms of give and take. Instructional environments beneficial to some students may disadvantage and be counterproductive to others. What mix of which types of learning environments is better suited for whom?

**Symbol sense.** Increased emphasis on symbolic operation carries with it the potential for increased mystification when students have no understanding of a displayed result. How many students who differentiate \( y = x^n \) can verify that their answer agrees with a CAS displayed result \( y' = n e^{\ln x} / x \) or \( y' = e^{n \ln x} n \ln e / x \) (based on using \( y = e^{\ln x} \))? What kinds of symbol sense should we expect of students? How can we help them discover patterns and relations that are masked by symbolic or
numerical displays (decimal representations)? What danger is there that the increased emphasis on push-button symbolic computations may lead students to view mathematics as operating with symbols on computer keys?

Solution verification and reconciled representations. To what extent should students be able logically to justify or computationally to verify a symbolic or numerical solution? Should we expect students to reconcile multiple representations of a result computed by different methods? Compare, for example

$$\int \sin^3 x \, dx = \frac{\sin^3 x}{3} - \frac{2 \cos x}{3} \quad [\text{Maple 4.1}], \quad = -\frac{3 \cos x}{4} + \frac{\cos 3x}{12} \quad [\text{Mathematica}],$$

$$= -\cos x + \frac{\cos^3 x}{3} \quad [\text{CAL 2.0}]$$

Mathematical exploration. Students engaged in technology-based self-directed explorations may acquire interesting mathematical results or interesting mathematical misconceptions. How much structure and guidance is appropriate for which students to engage in what kinds of explorations? What higher-order knowledge do students need to know when they have discovered something important and what makes it so?

Syllabus-student interface. Much remains unknown about the role and value that routine processes and computations play in learning mathematics. Does proficiency with straightforward, mechanical processes give students satisfaction or motivation to stay vested and continue coping with harder, more varied subject matter? How much routine, repetitive material is needed to provide students with intellectual rest stops before driving on to more demanding mathematical terrain? How will more concept-oriented, experientially based courses change our definition of what constitute remediation? Many of the classes we teach are made up of students with diverse career objectives. Thus, they may have differing predispositions towards various kinds of mathematical knowledge and ways of using it. How do we accommodate the different cognitive and affective needs of students within the same class?

Knowledge assessment. What we assess and how we do so influences students’ perceptions of what is expected and how they are meeting these expectations. Although we stress understanding and critical thinking, our test questions often stress routine processes and computations exemplified by the verbs solve, sketch, evaluate, calculate, compute, differentiate, integrate, invert, etc. – precisely those mindless processes keyed to what CASs can do best. Asking students to “state and prove” also reveals no understanding of what may have been memorized. To reduce the disparity between our educational goals and assessment objectives, we need to use examination questions characterized by the verbs define and illustrate, explain, describe, compare, justify, interpret, etc. Fifteen or twenty minutes of probing discourse with a student at a computer could reveal a great deal about the student’s subject specific knowledge, as well as help us gauge what students have gained from our courses by observing what they do when they do not know the answers to posed problems. In what ways will technology alter the format, administration, diagnostic or predictive utility of standardized examinations?
Impact on faculty and mathematics departments. Career development research shows that the priorities, commitments, interests, and abilities vary considerably among teaching assistants, adjuncts, beginning college instructors, professors at mid-career, and faculty nearing retirement. Thus, it may not be reasonable to expect all faculty to be willing or able to stay current in the instructional uses of evolving technologies. Which populations of teachers are or will be at risk as technology assumes greater prominence in mathematics instruction? What does this portend for the pool of people that can be hired as part-time mathematics instructors? Successful curriculum innovation requires the continuing, broadly supported efforts of a mathematics department, as opposed to ad hoc approaches where a few colleagues each use different software, and others use none at all. What are the responsibilities of a Mathematics Department to its students? To its faculty?

REFERENCES


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